STABILITY OF MOTION UNDER IMPULSIVE PERTURBATIONS

BY

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DEPARTMENT OF MATHEMATICS
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STABILITY OF MOTION UNDER IMPULSIVE PERTURBATIONS

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BY

VADREVU SREE HARI RAO

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Dedicated

to

My Beloved Parents

Smt. V. Satyavathi

and

(Late) Sri V. Seshagiri Rao

without whose blessings this work would not have been possible.

CERTIFICATE

This is to certify that the work embodied in the thesis entitled "Stability of Motion Under Impulsive Perturbations" being submitted by Vadrevu Sree Hari Rao has been carried out under my supervision and that this has not been submitted elsewhere for the award of any degree.

July, 1976.

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(V. SREE HARI RAO)

July, 1976.

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CONTENTS

SYNOPSIS

A thesis entitled "Stability of Motion under Impulsive Perturbations", is submitted in partial fulfilment of the requirements for the Ph.D. degree by Vadrevu Sree Hari Rao to the Department of Mathematics, Indian Institute of Technology, Kanpur.

In a natural way, nonlinear differential and integral equations arise in diverse areas such as physical, biological, oceanographic and engineering sciences. The way such equations arise and the properties of solutions of these equations for various physical phenomena have been investigated by many scientists. However until quite recently, the attempts have not been made to develop and unify the theory of perturbations such as the theory of ordinary differential equations with impulsive perturbations and Volterra integral equations with discontinuous perturbations. The objective of this investigation is to study the stability and boundedness properties of solutions of ordinary differential equations with impulsive perturbations, Volterra integro-differential equations containing measures and nonlinear mixed integral equations of Volterra-Fredholm type involving Lebesgue-Stieltjes integrals.

In most of the physical systems such as aircrafts and space rockets, some external forces act directly or indirectly. Before formulating a mathematical model, it is necessary to anticipate how big these external forces could be tolerated to keep the system

stable for all the future time. In case, these external forces are not continuous with respect to time, which is quite common in nature, the problem becomes more intricated. Naturally such systems contain impulses and handling them needs more sophisticated recently developed tools of modern analysis. Not only physical systems but also biological systems like heart beats, blood flows, pulse frequency modulations and neural nets may probably exhibit the same phenomenon. In a more general situation, systems with impulsive perturbations, in fact, give rise to equations of the form

$$Dx = F(t,x) + G(t,x)\,Du \tag{1}$$

where Du denotes the distributional derivative of u. If u is a function of bounded variation, then Du can be identified with Stieltjes measure and will have the effect of instantaneously changing the state of the system at the points of discontinuity of u. For example, system (1) with $F(t,x) = Ax$, $G(t,x) = 1$ and $u(t) = u_o(t) + \sum_{k=1}^{\infty} a_k\,\delta(t-t_k)$, where $u_o(t)$ is a locally integrable function, A an $n \times n$ constant matrix, a_k the elements of R^n and $\delta(t - t_k)$ is the Dirac function, may be regarded as a link in an automatic control chain transmitting the distribution associated with the function u. System (1) may also be treated as a perturbed system of the ordinary differential system

$$x' = F(t,x) \tag{2}$$

where the perturbation $G(t,x)\,Du$ is impulsive. In this set up the following question would be interesting and highly desirable :

under what conditions on the perturbations, the stability and boundedness properties of solutions of (2) are shared by the corresponding properties of solutions of (1). To get a satisfactory answer to this question is rather difficult. It may be so since differential and integral inequalities play a crucial role in stability theory of ordinary differential equations. As the solutions of (1) are discontinuous, most of the differential inequalities are not applicable while the integral inequalities for Stieltjes integrals are not available. However, attempts have been made in this thesis to obtain satisfactory answer to the above said question, for a variety of classes of perturbations with less restrictive conditions. Thus our results improve and include the corresponding results of ordinary differential equations under continuous perturbations.

The present thesis is divided into five chapters. The first chapter is devoted for introduction and outline of the thesis. Chapter 2 deals with preliminaries and basic results such as ideas of Liapunov's second method, distributions and distributional derivatives, and the integral representation of equation (1).

The objective of chapter 3 is to discuss stability and boundedness properties of solutions of (1). In sections 3.2, 3.3 a special case of system (1), namely

$$Dx = F(t,x) + p(t)\, Du \tag{3}$$

is considered and obtained sufficient conditions for integral and integral asymptotic stability of the zero solution of (2) and the

uniform asymptotic stability of zero solution of (2) under interval bounded perturbations (Definition is given in the text). Uniform and uniform ultimate boundedness of solutions of (3) are also discussed in section 3.4, assuming the corresponding properties of (2). In all these three sections, the perturbation $p(t)\,Du$ in system (3) could have been replaced by a perturbation $G(t,x)\,Du$ which satisfies the condition $|G(t,x)| \leq |p(t)|$ for all x in some neighbourhood of the origin. Since this does not introduce any new ideas, we confine to the disoussion of system (3). However, the corresponding changes are indicated in the case of system (1). In the rest of the sections, eventual uniform asymptotic stability of zero solution of (1) and the asymptotic equivalence between (1) and (2) have been studied. Simple examples are constructed to illustrate the fruitfulness of the results.

In the first part of chapter 4, we consider a functional differential equation of delay type in which the interval of delay is $[0,\bar{t}]$, $t > 0$. In particular, we include integro-differential equations of Volterra type

$$x' = F(t,x) + \int_0^t K(t,s,x(s))\,ds \tag{4}$$

where $F : J \times R^n \to R^n$, $K : J \times J \times R^n \to R^n$, $J = [0,\infty)$.

By using Liapunov-Razumikhin conditions, sufficient conditions for uniform and exponential asymptotic stability of zero solution of such equations are investigated. In the latter part, we consider a more general type of integro-differential system involving measures,

namely

$$Dx = F(t,x) + G(t,x)\ Du + \int_0^t K(t,s,x(s))\ ds \qquad (5)$$

where $G : J \times R^n \to R^n$ and $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J and obtained sufficient conditions for eventual uniform asymptotic stability of zero solution of (5). A few examples are given and the intricacies that arise due to the presence of $G(t,x)\ Du$ in equation (5) are indicated.

Finally, chapter 5 is concerned with the study of existence, uniqueness and stability properties of solutions of Volterra-Fredholm system

$$x(t) = f(t) + \int_0^t a(t,s)\ g(s,x(s))\ du(s) + \int_0^\infty b(t,s)h(s,x(s))\ dw(s) \qquad (6)$$

where x, f, g, h are n-vectors, a and b are $n \times n$ matrices, u and w are right continuous functions of bounded variation on every compact sub-interval of J. The conditions that are used to derive the above properties of solutions of (6) are some what restrictive which are natural to expect because of discontinuous nature of solutions of (6). However, some less restrictive conditions are indicated and their usefulness is discussed through examples so as to include the case of convolution kernels.

CHAPTER - 1

INTRODUCTION

1.1 Historical Notes

The theory of stability of motion has gained increasing significance in the last two decades as is apparent from the large number of publications on this subject. A considerable part of this work is concerned with practical problems especially from the area of controls and servo-mechanisms and concrete problems from engineering. These problems are instrumental for giving decisive impetus for the expansion and modern development of stability theory.

Basically there are two fundamental concepts in stability theory : stability in the sense of Liapunov, i.e., stability with respect to change of initial conditions; and total stability (in the Soviet terminology stability with respect to constantly acting perturbations). Liapunov stability for linear, nonlinear and perturbed linear and nonlinear systems have been considered by various authors and excellent monographs have been published. We refer the reader to Bellman [3] , Coppel [7] , Hahn [19] , Yoshizawa [61] , and Lakshmikantham and Leela [23] and for total stability Yoshizawa [61] .

Among others, Vrkoc [59,60] , Chow and Yorke [55] , Strauss and Yorke [56,57] , Rama Mohana Rao and Tsokos [46] ,

Lakshmikantham and Rama Mohana Rao [24] , Seifert [16,17] have studied stability of solutions of perturbed nonlinear differential systems under various classes of perturbations and many interesting results have been accumulated.

When a physical system described by an ordinary differential equation $\frac{dx}{dt} = F(t,x)$ is subject to perturbations, the perturbed system is generally given by an ordinary differential equation $\frac{dx}{dt} = F(t,x) + G(t,x)$ in which the perturbation function $G(t,x)$ is assumed to be continuous or integrable. Obviously in this case the state of the system changes continuously with respect to time. It is natural to expect that in most of the physical systems, the perturbations need not be continuous or integrable (in the usual sense); thus the state of the system changes discontinuously with respect to time and hence the study of stability of systems with impulsive perturbations is highly desirable and more meaningful than ever before.

1.2 Brief Review

In most of the physical systems such as air crafts and space rockets, some external forces act directly or indirectly. Before we design and build a mathematical model, it is natural to anticipate how big these external forces could be tolerated to keep the system stable for all the future time. In case these external forces are not continuous with respect to time, which is quite common in nature, the problem becomes more complicated. Naturally such systems contain impulses and handling them needs more sophisticated recently developed

tools of modern analysis. For example, consider the following system

$$x' = Ax + u \tag{1.2.1}$$

where x and u are vectors in R^n and A is an $n \times n$ matrix. In (1.2.1) u includes functions of the type

$$u(t) = u_o(t) + \sum_{i=1}^{\infty} a_i \, \delta(t - t_i) \tag{1.2.2}$$

where $u_o(t)$ is a locally integrable function, a_i the elements of R^n and $\delta(t - t_i)$ is the Dirac function, i.e., the simplest generalized function concentrated at the point t_i. Here system (1.2.1) is regarded as a link in an automatic control chain transmitting the distribution associated with the function (1.2.2). In connection with this, it is interesting to study questions of stability in the case, more general than (1.2.2), when the input function u(t) is allowed to be an arbitrary generalized function. It is clear that the solutions of this problem would make it possible to develop a theory encompassing classical stability theory from a unified point of view. In addition, the necessity of studying a series of similar problems is dictated by the fact (see Koshlyakov [21]) that the description of physical processes in the language of set functions, distributions and, in particular, generalized functions, gives a more accurate reflection of the real nature of these processes. Not only physical systems but also biological systems like heart beats, blood flows, pulse frequency modulations and neural nets (see Pavlidis [42,43] and the references contained there in) may probably exhibit the same phenomenon.

In a more general set up such systems in fact give rise to equations of the form

$$Dx = F(t,x) + G(t,x)\, Du \tag{1.2.3}$$

where Du denotes the distributional derivative of the function u. If u is a function of bounded variation, then Du can be identified with Stieltjes measure and will have the effect of suddenly changing the state of the system at the points of discontinuity of u. Equations of the form (1.2.3) are generally known as ordinary differential equations with impulsive perturbations or measure differential equations. The equation (1.2.3) may be regarded as a perturbed system of the ordinary differential system

$$x' = F(t,x) \tag{1.2.4}$$

where $G(t,x)\, Du$ represents an impulsive perturbation. A natural question that arises in this connection is that: under what conditions on the perturbations $G(t,x)\, Du$ the stability and boundedness properties of solutions of (1.2.4) are shared by the solutions of (1.2.3). It seems rather difficult to get a satisfactory answer to this question. It may be so because differential and integral inequalities play a vital role in the stability theory. Since the solutions of (1.2.3) are discontinuous, most of the differential inequalities are not applicable where as integral inequalities for Stieltjes integrals are not available. The stability of systems (1.2.1) and (1.2.2) are studied by Barbashin [2] and Zavalishchin [63] . Quite recently, among others, Das and Sharma [13,14] and Raghavendra and Rama Mohana Rao [44,45] have considered equation (1.2.3) and obtained sufficient

conditions for eventual uniform asymptotic stability of zero solution of (1.2.3) under some what restrictive conditions.

The attempts have been made in this thesis to study the problems of stability and boundedness of solutions of (1.2.3) under less restrictive conditions for a variety of perturbation functions G(t,x) Du. Motivated by the work of Seifert [16,17] , integro-differential equations of Volterra type are considered as special cases of functional differential equations of Volterra type (FDEVT) with infinite delay and obtained sufficient conditions for uniform and exponential asymptotic stability of zero solution of FDEVT by employing Liapunov-Razumikhin conditions. Furthermore, integro-differential equations of Volterra type containing measures are also discussed. Nonlinear integral equations of Volterra-Fredholm type have been considered by Miller, Nohel and Wong [37] in 1969. Inspired by their work, nonlinear mixed integral equations involving Lebesgue-Stieltjes integrals are also discussed. Our study includes some of the results of [23] , [37] , [55] , [56] , [61] . The fruitfulness of the results is illustrated with examples and applications.

1.3 Layout of the Thesis

The purpose of the present thesis is to study the problem of stability of motion for ordinary differential equations with respect to impulsive perturbations, certain class of Volterra integro-differential equations and nonlinear mixed integral equations of Volterra-Fredholm type.

The work embodied in this thesis is divided into five chapters and each chapter is further subdivided into sections. Chapters 3,4 and 5 constitute the main core of the thesis.

Chapter 1 is concerned with introduction, history, motivation and relevance of the problem of stability of motion under consideration.

Chapter 2 is exclusively devoted for the preliminaries and basic results and thus provides necessary input to the main results of the thesis.

Chapter 3 aims at discussing the stability and boundedness of solutions of ordinary differential systems with respect to impulsive perturbations. More specifically sufficient conditions for integral stability, integral asymptotic stability, uniform asymptotic stability under a more general class of perturbations namely interval bounded perturbations, uniform and uniform ultimate boundedness and eventual uniform asymptotic stability are obtained and lastly asymptotic equivalence between (1.2.3) and (1.2.4) is discussed. Our results improve and include the corresponding results on ordinary differential equations with continuous perturbations. Some examples are provided to illustrate the fruitfulness of the results.

Integro-differential equations of Volterra type containing measures are discussed in chapter 4. In the absence of measures these equations are considered as special cases of functional differential equations of Volterra type with infinite delay and some interesting stability results are obtained by using Liapunov-Razumikhin conditions. Furthermore, for the equations involving

measures, the Liapunov's second method is extended and obtained sufficient conditions for eventual uniform asymptotic stability of zero solution. Number of examples are given in both the cases and the intricacies that arise due to the presence of measures in the latter case are indicated.

Lastly in chapter 5, the existence, uniqueness and stability properties of solutions of nonlinear mixed integral equations of Volterra-Fredholm type involving Lebesgue-Stieltjes integrals are considered under some what restrictive conditions. Furthermore, some less restrictive conditions are given in the text of the section 5.3 and are discussed through some illustrative examples. In the end, an interesting problem has been posed so as to include the case of convolution kernels.

CHAPTER - 2

PRELIMINARIES AND BASIC RESULTS

2.1 Introduction

The theory of differential and integral equations plays a vital role in physical, biological, social and engineering systems which are dynamic in character and therefore the solutions of these equations must come down to explicit or numerical expressions. However, all too often this can only be accomplished under very restrictive approximations. The problem therefore arises to obtain some qualitative information about the elusive solutions. There comes the qualitative examination of the systems behaviour such as stability, boundedness, periodicity and oscillatory properties of solutions. In this section, we give the basic ideas of a method known as Liapunov's second method to determine the stability and boundedness of such systems which can be applied directly to the systems concerned without the explicit knowledge of solutions.

The true creator of stability theory is A.M. Liapunov and the starting point for modern theories is his famous mémoire published in Russian in 1892 and French translation in 1907 "Problème générale de la stabilité de mouvement". Although Liapunov introduced a method which is called Liapunov's second method and used it only to establish stability theorems, this method has been widely recognized today as an indispensible tool not only in the theory of stability but also in studying many other qualitative

properties of solutions of ordinary differential systems and integro-differential systems of Volterra type. The main characteristic of his approach is the construction of a scalar function namely Liapunov function $V(t,x)$ and his method crucially depends on the sign of the total derivative of $V(t,x)$ along the solutions of the given system under consideration. This method is also known as Liapunov's direct method since these techniques can be applied directly to the differential system without any prior knowledge of solutions. Excellent accounts of this method for linear and nonlinear ordinary differential systems with continuous perturbations may be found in Antosiewicz [1] , Hahn [19] , Krasovskii [22] , Lakshmikantham and Leela [23] , La Salle and Lefschetz [25] and Yoshizawa [61] . The main object of this thesis is to extend Liapunov's second method to study the stability and boundedness properties of solutions of ordinary differential equations with impulsive perturbations and integro-differential equations of Volterra type containing measures.

2.2 Definitions and Fundamental Theorems

This section provides all the necessary definitions and basic theorems which will be used throughout the text of the thesis. In particular, since converse theorems play an important role in studying the properties of solutions of perturbed systems, such theorems for (1.2.4) find a place in this section.

Let $x(t,t_o,x_o)$ be any solution of the differential system (1.2.4) with $x(t_o) = x_o$, $t_o \geq 0$ and $|x| = \sum_{i=1}^{n} |x_i|$, where

$F : J \times S_\rho \to R^n$, is continuous and sufficiently smooth on $J \times S_\rho$, S_ρ being the set, $S_\rho = \{x \in R^n : |x| < \rho\}$ and $J = [0,\infty)$. For any square matrix A, the norm of A is defined by $|A| = \sum_{i,j=1}^{n} |a_{ij}|$.

Assume that $F(t,0) = 0$, for $t \in J$ so that $x \equiv 0$ is a (trivial) solution of (1.2.4) through $(t_0,0)$.

Definition 2.2.1.

The trivial solution $x \equiv 0$ of (1.2.4) is

(i) uniformly stable if, for each $\varepsilon > 0$, $t_0 \in J$, there exists a positive number $\delta = \delta(\varepsilon) > 0$ such that the inequality $|x_0| < \delta$ implies $|x(t,t_0,x_0)| < \varepsilon$, for all $t \geq t_0$;

(ii) uniformly asymptotically stable if it is uniformly stable and in addition there exists $\delta_0 > 0$ and for each $\eta > 0$, $t_0 \in J$, there exists a $T = T(\eta) > 0$ such that $|x_0| < \delta_0$ implies $|x(t,t_0,x_0)| < \eta$, for all $t \geq t_0 + T$.

Definition 2.2.2.

A function $\phi : [0,\rho) \to [0,\infty)$ is said to belong to the class K if $\phi(r)$ is continuous, $\phi(0) = 0$ and $\phi(r)$ is strictly increasing in r.

Definition 2.2.3.

A function $V(t,x)$ with $V(t,0) \equiv 0$ is said to be positive definite (negative definite) if there exists a function $\phi(r) \in K$ such that the relation

$$V(t,x) \geq \phi(|x|) \qquad (\leq -\phi(|x|))$$

is satisfied for $(t,x) \in J \times S_\rho$.

Definition 2.2.4.

A function $V(t,x) \geq 0$ is said to be decrescent if a function $\phi(r) \in K$ exists such that

$$V(t,x) \leq \phi(|x|), \quad (t,x) \in J \times S_\rho.$$

Definition 2.2.5.

$\bar{C}_0(x)$ denotes the class of functions having uniform Lipschitz constants with respect to x on $J \times S_\rho$; C_m the class of functions having continuous partial derivatives of order $k = 1,2,\dots,m$; and $\bar{C}_\infty$ the class of functions having continuous and bounded partial derivatives of every order on $J \times S_\rho$.

Definition 2.2.6.

A scalar function $V(t,x)$ is a Liapunov function for (1.2.4) if

(i) $V(t,x)$ is positive definite and C_1 on $J \times S_\rho$,

(ii) $V(t,0) \equiv 0$,

(iii) $V'(t,x) \equiv \frac{\partial}{\partial t} V(t,x) + \nabla V(t,x) \cdot F(t,x) \leq 0$

for $(t,x) \in J \times S_\rho$ where $\nabla V = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n})$.

We need the following converse theorem whose proof can be found in [33] .

Theorem 2.2.1 (Massera [33]).

If, for (1.2.4), $F \in \bar{C}_0(x)$ on $J \times S_\rho$ and $x \equiv 0$ is uniformly asymptotically stable, then there exists a Liapunov function $V(t,x)$ on $J \times S_\rho$ for (1.2.4) such that $V(t,x)$ is positive definite, decrescent and $V'(t,x)$ is negative definite and $V \in \bar{C}_\infty$.

2.3 Distributions and Distributional Derivatives

We consider a function x on the set of real numbers and taking values in R^n. We consider all possible partitions of an interval $[a,b]$ in the real numbers into a finite number of subintervals : $a = t_0 < t_1 < \ldots < t_n = b$. The quantity

$$\sup \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})|$$

where the upper bound is taken over all possible partitions of the interval $[a,b]$, is called (by definition) the total variation of the function x on the interval $[a,b]$ and is denoted by $V(x,[a,b])$. We shall say that x is of bounded variation on $[a,\infty)$, if x has bounded variation on any interval $[a,t]$, $a \leq t < \infty$, and the set of total variations $V(x, [a,t])$ is bounded. By definition, we have

$$V(x, [a,\infty)) = \sup_{t \geq a} V(x, [a,t]).$$

$BV(J,R^n) = BV(J)$ = the space of all functions of bounded variation defined on J and taking values in R^n. The norm of $x \in BV(J)$ is defined by

$$||x|| = V(x,J) + |x(0^+)|,$$

where 0 is the left end point of J. With this norm, BV(J) is a Banach space.

Let Ω be an open subset of R^n. We denote by $C_c^\infty(\Omega)$, the class of complex valued functions defined on Ω , whose support is compact and which have partial derivatives of all orders $< \infty$. $C_c^\infty(\Omega)$ is a normed linear space with addition, scalar multiplication and norm

defined by

$$(\phi_1 + \phi_2)(x) = \phi_1(x) + \phi_2(x), \quad \phi_1, \phi_2 \in C_c^\infty(\Omega)$$

$$(\alpha\phi)(x) = \alpha\phi(x) \qquad , \quad \phi \in C_c^\infty(\Omega)$$

$$|\phi| = \sup_{x \in \Omega} |\phi(x)|.$$

A continuous linear functional defined on $C_c^\infty(\Omega)$ is called a distribution on Ω. It follows from Riesz representation theorem that the set of all complex Borel measures on Ω is, by $\mu \leftrightarrow T_\mu$, in one-to-one correspondence with the set of all distributions on Ω, where T_μ is the distribution defined by

$$T_\mu(\phi) = \int_\Omega \phi \, d\mu , \qquad \phi \in C_c^\infty(\Omega).$$

Let a complex function f defined a.e. on Ω be locally integrable on Ω with respect to the Lebesgue measure in the sense that for any compact subset K of Ω, $\int_K |f(x)| \, dx < \infty$. Then

$$T_f(\phi) = \int_\Omega f(x)\, \phi(x) \, dx, \quad \phi \in C_c^\infty(\Omega),$$

defines a distribution T_f on Ω. Two distributions T_{f_1} and T_{f_2} are equal as functionals ($T_{f_1}(\phi) = T_{f_2}(\phi)$ for every $\phi \in C_c^\infty(\Omega)$) if and only if $f_1(x) = f_2(x)$ a.e. (see [62], p. 48).

The derivative of a distribution T with respect to x^i, denoted by $D_i T$ or $\frac{\partial T}{\partial x^i}$, is defined by

$$D_i T(\phi) = -T\left(\frac{\partial \phi}{\partial x^i}\right), \quad \phi \in C_c^\infty(\Omega),$$

and is also a distribution on Ω. A distribution is infinitely differentiable in the sense of above definition.

Since a locally integrable function f on an open interval I of real line can be identified with the distribution T_f on I, $DT_f (\equiv \frac{dT_f}{dt})$ will be denoted by Df and called distributional derivative of f to distinguish from its ordinary derivative $f' (\equiv \frac{df}{dt})$. If f is absolutely continuous, then Df is the ordinary derivative f' (which is defined a.e.), f' being considered equivalent to the distribution $T_{f'}$. If f is a function of bounded variation then Df is the Lebesgue-Stieltjes measure df.

2.4 Integral Representation of Measure Differential Equations

Consider the measure differential equation

$$Dx = F(t,x) + G(t,x)\, Du \tag{2.4.1}$$

where F and G are defined on $J \times R^n$ with values in R^n and $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J.

Let I be an interval with left end point $t_0 \geq 0$.

Definition 2.4.1.

A function $x(t) = x(t,t_0,x_0)$ is said to be a solution of (2.4.1) through (t_0,x_0) on I if x is right continuous and $x(t) \in BV(I,R^n)$, $x(t_0) = x_0$ and the distributional derivative of x on $[t_0, t_0+\alpha]$ for every $\alpha \in I$ satisfies (2.4.1).

The following lemma is used to prove the integral representation of (2.4.1), whose proof can be found in [14] .

Lemma 2.4.1.

If g is a function integrable with respect to μ , and T is a distribution on Ω given by

$$T(\phi) = \int_{\Omega} \phi \, d\mu \ , \quad \phi \in C_c^{\infty}(\Omega),$$

then the product gT defined by

$$(gT)\,(\phi) = \int_{\Omega} g \, \phi \, d\mu \, , \quad \phi \in C_c^{\infty}(\Omega),$$

is also a distribution on Ω.

Now, consider the integral equation

$$x(t) = x_o + \int_{t_o}^{t} F(s,x(s))\,ds + \int_{t_o}^{t} G(s,x(s))\,du(s) \qquad (2.4.2)$$

Theorem 2.4.1.

$x(t)$ is a solution of (2.4.1) through (t_o,x_o) on an interval $I = [t_o,t_o+b]$, if and only if $x(t)$ satisfies (2.4.2) for $t \in I$.

Proof.

Let $x(t)$ satisfy (2.4.2) for $t \in I$. The integral $\int_{t_o}^{t} F(s,x(s))ds$ is absolutely continuous (hence continuous and of bounded variation) function of t on I. The integral $\int_{t_o}^{t} G(s,x(s))du(s)$ is a function of bounded variation on I and the right continuity of u implies that it is also a right continuous function of t. Therefore

$x(t) \in BV(I,R^n)$ and is right continuous. Obviously $x(t_o) = x_o$. Let a be any arbitrary point in I and let T^i be the distribution on $[t_o,a]$ which is to be identified with the ith component $x^i(t)$ of $x(t)$. Then

$$T^i(\phi) = \int_{J_1} [x_o^i + \int_{t_o}^{t} F^i(s,x(s))ds + \int_{t_o}^{t} G^i(s,x(s))du(s)]\ \phi(t)dt \quad (2.4.3)$$

for all $\phi \in C_c^\infty(J_1)$ where $J_1 = [t_o,a] \subseteq I$. The distributional derivative is

$$DT^i(\phi) = -T^i(\phi') = -\int_{J_1} [x_o^i + \int_{t_o}^{t} F^i(s,x(s))ds + \int_{t_o}^{t} G^i(s,x(s))du(s)]\phi'(t)dt \quad (2.4.4)$$

Integration by parts yields

$$-\int_{J_1} [x_o^i + \int_{t_o}^{t} F^i(s,x(s))ds]\ \phi'(t)\ dt = \int_{J_1} \phi(t)\ F^i(t,x(t))dt \quad (2.4.5)$$

since $\phi(t_o) = \phi(a) = 0$. The function $g(t) = \int_{t_o}^{t} G^i(s,x(s))du(s)$ is right continuous and is of bounded variation on the interval J_1 by [34, §52.18, p. 275]. We have

$$\int_{J_1} g(t)\ \phi'(t)\ dt = g(a)\ \phi(a) - g(t_o)\ \ \phi(t_o) - \int_{J_1} \phi(t)\ dg(t)$$

$$= -\int_{J_1} \phi(t)dg(t), \text{ since } \phi(t_o) = \phi(a) = 0.$$

That is

$$\int_{J_1} \{\int_{t_o}^{t} G^i(s,x(s))\ du(s)\}\ \phi'(t)\ dt$$

$$= -\int_{J_1} \phi(t)\ d\ \{\int_{t_o}^{t} G^i(s,x(s))\ du(s)\}\ .$$

$$= -\int_{J_1} \phi(t)\ G^i(t,x(t))du(t), \text{(cf. [15] , corollary 6,p. 180)}. \quad (2.4.6)$$

From (2.4.4), (2.4.5) and (2.4.6), we obtain

$$DT^i(\phi) = \int_{J_1} \phi(t)F^i(t,x(t))dt + \int_{J_1} \phi(t)\, G^i(t,x(t))du(t). \qquad (2.4.7)$$

By lemma 2.4.1, the last continuous linear functional in (2.4.7) is identified with the measure $G^i(t,x(t))\, du(t)$, while the first continuous linear functional in (2.4.7) is identified with $F^i(t,x(t))$. This holds for $i = 1,2,\ldots,n$, and therefore the derivative distribution $DT(\phi)$ is identified with $F(t,x(t)) + G(t,x(t))\, Du$. Hence $x(t)$ is a solution of (2.4.1) through (t_o,x_o).

Conversely, let $x(t)$ be a solution of (2.4.1) through (t_o,x_o) on I. Then we have

$$\int_{J_1} \phi(t)\, Dx^i(t) = \int_{J_1} \phi(t)\, F^i(t,x(t))dt + \int_{J_1} \phi(t)\, G^i(t,x(t))\, du(t)$$

$$i = 1,2,\ldots,n. \qquad (2.4.8)$$

for all $\phi \in C_c^\infty(J_1)$. By using [15, corollary 6, p. 180] again we may write

$$\int_{J_1} \phi(t)\, G^i(t,x(t))\, du(t) = \int_{J_1} \phi(t)\, d\, \{\int_{t_o}^{t} G^i(s,x(s))\, du(s)\}\, .$$

Integrating by parts the integrals in (2.4.8) in the way we have done above, we obtain

$$\int_{J_1} \phi'(t)(x^i(t)-x^i(t_o))dt = \int_{J_1} \phi'(t)\, [\int_{t_o}^{t} F^i(s,x(s))ds + \int_{t_o}^{t} G^i(s,x(s))du(s)]\, dt.$$

Therefore,

$$x^i(t) = x^i(t_o) + \int_{t_o}^{t} F^i(s,x(s))\, ds + \int_{t_o}^{t} G^i(s,x(s))\, du(s) \quad (2.4.9)$$

a.e. in J_1. But, since $x^i(t)$ is continuous from the right, being a solution of (2.4.1) and since the right hand side of (2.4.9) is a right continuous function of t, equality holds everywhere in J_1 for (2.4.9). Hence x(t) is a solution of (2.4.2). This completes the proof.

For completeness, lastly we state the following existence and uniqueness theorem for the system (2.4.1) whose proof is exactly similar to that of theorem 1 in [14] .

Theorem 2.4.2.

Suppose F and G satisfy the following conditions on the set

$$E = \{(t,x);\ t \in [t_o, t_o+b]\ ,\ x \in S_r\}$$

(i) F is continuous in x for each fixed t and measurable in t for each fixed x and satisfies a local Lipschitz condition in x.

(ii) there exists a Lebesgue integrable function m, such that

$$|F(t,x)| \leq m(t),\ (t,x) \in E\ ;$$

(iii) G(t,x) is du-integrable for each $x(\cdot) \in BV([t_o,t_o+b]\ ,\ S_r)$;

(iv) G(t,x) is continuous in x for each fixed t ;

(v) there exists a dv_u -integrable function ω such that

$$|G(t,x)| \leq \omega(t),\ \ (t,x) \in E\ ;$$

where v_u-denotes the total variation function of u. Then there exists a unique solution $x(\cdot)$ of (2.4.1) on some interval $[t_o, t_o+\beta]$, $\beta < b$, satisfying the initial condition $x(t_o) = x_o \in S_r$.

CHAPTER - 3

STABILITY AND BOUNDEDNESS OF SOLUTIONS OF IMPULSIVELY PERTURBED NONLINEAR DIFFERENTIAL SYSTEMS

3.1 Introduction

The study of perturbation problems for ordinary differential equations is one of the fruitful areas of research and a great deal of work has been done by many authors in recent years. Historically, there have been two approaches to these problems. The first is to set conditions on the unperturbed system and find out the type of perturbations, that preserve the stability behaviour of the unperturbed system (cf. [6, chap. 13], [7], [23], [56], [57], [61]). The second approach is to set the kind of perturbations that will be allowed and find the differential equations whose stability and boundedness properties are preserved by those perturbations (cf. [18], [32], [59], [60]). Motivated from the works of Vrkoc [59], Okamura [41] and Yoshizawa [61], quite recently Chow and Yorke [55] gave necessary and sufficient conditions for integral and integral asymptotic stability of the unperturbed system and showed that every integrally asymptotically stable system behaves nicely not only for perturbations integrable on $[0,\infty)$ but also for a larger class of perturbations such as interval bounded perturbations.

When a physical system described by an ordinary differential equation, is acted upon by certain perturbations, the perturbed system is again an ordinary differential equation in which the perturbation

functions are assumed to be continuous or integrable. Clearly in this case the state of the system changes continuously with respect to time. However in most of the physical systems, the perturbation functions need not be continuous or integrable (in the usual sense) and thus the state of the system changes discontinuously with respect to time. Indeed, the study of stability and boundedness properties of solutions of impulsively perturbed systems is dictated by the fact that the description of the physical processes in the language of set functions, distributions and, in particular, generalized functions, gives a more accurate reflection of the real nature of these processes.

The first part of this chapter deals with the second approach while the latter part deals with the first approach for impulsively perturbed nonlinear ordinary differential systems.

3.2 Integral and Integral Asymptotic Stability

In this section we study the integral and integral asymptotic stability of the trivial solution $x \equiv 0$ of

$$x' = F(t,x) \tag{3.2.1}$$

where $F : J \times R^n \to R^n$ is a continuous function. Now suppose one knows that all the solutions of (3.2.1) which start near $x \equiv 0$ remain near this solution for all future time, or even that they approach this solution as time increases. If the differential equation (3.2.1) is acted upon by certain impulsive perturbations, the above property concerning the solutions near $x \equiv 0$ may or may not be true. More

precisely, if the trivial solution of (3.2.1) is asymptotically stable and if the function $p(t)$ Du is small in certain sense, then give conditions on F so that the trivial solution $x \equiv 0$ of (3.2.1) is asymptotically stable with respect to the perturbed system

$$Dx = F(t,x) + p(t)\,Du \qquad (3.2.2)$$

where $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J and $p : J \to R^n$ is integrable with respect to u and the discontinuities $t_0 < t_1 < t_2 < t_3 < \cdots < t_k < \cdots$ of u tend to ∞ as k tends to ∞.

The proofs of our results in this section and also in the subsequent sections of this chapter, crucially depend on almost everywhere differentiability of the function u and this property is guaranteed because u is a function of bounded variation. In fact, a function of bounded variation has a finite differential coefficient almost everywhere ([58] , p. 356). Throughout we denote by v_u, the total variation function of u.

Let $x(t,t_0,x_0)$ be any solution of (3.2.2) through (t_0,x_0) existing to the right of $t_0 \geq 0$.

Definition 3.2.1.

The trivial solution $x \equiv 0$ of (3.2.1) is said to be integrally stable, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x_0| < \delta$ and $\int_{t_0}^{\infty} |p(s)|\, dv_u(s) < \delta$ imply $|x(t)| < \varepsilon$, for all $|x_0| < \delta$ and $t \geq t_0 \geq 0$.

Definition 3.2.2.

The trivial solution $x \equiv 0$ of (3.2.1) is said to be integrally attracting, if there exists a $\delta_0 > 0$ and for each $\eta > 0$ there exist $T = T(\eta) > 0$ and $\alpha = \alpha(\eta) > 0$ such that $|x_0| < \delta_0$ and $\int_{t_0}^{\infty} |p(s)|\, dv_u(s) < \alpha$ imply $|x(t)| < \varepsilon$, for all $t \geq t_0 + T$ and $t_0 \geq 0$.

Definition 3.2.3.

The trivial solution $x \equiv 0$ of (3.2.1) is said to be integrally asymptotically stable if the definitions 3.2.1 and 3.2.2 hold simultaneously.

Now we prove the following lemmas that are used throughout this chapter.

Lemma 3.2.1.

If a solution $x(t)$ of (3.2.2) exists and differentiable for $t \in [t_{k-1}, t_k)$, $k = 1,2,3,\ldots$, then the inequality

$$V'_{(3.2.2)}(t, x(t)) \leq V'_{(3.2.1)}(t, x(t)) + L\,|p(t)|\,|u'(t)|, \quad t \in [t_{k-1}, t_k)$$

$k = 1,2,3,\ldots$ where $|\nabla V(t,x)| \leq L$ and $\nabla V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right)$, holds.

Proof.

Since u is continuous and differentiable on $[t_{k-1}, t_k)$, $k = 1,2,3,\ldots$, as long as a solution $x(t)$ of (3.2.2) exists and differentiable for $t \in [t_{k-1}, t_k)$, we have

$$\frac{d}{dt} V(t,x(t)) \underset{(3.2.2)}{=} V'(t,x(t)) = \frac{\partial}{\partial t} V(t,x(t)) + \nabla V(t,x(t)).[F(t,x(t)) + p(t)\, u'(t)]$$

$$= \frac{\partial}{\partial t} V(t,x(t)) + \nabla V(t,x(t)) \,.\, F(t,x(t)) + \nabla V(t,x(t)) \,.\, p(t)\, u'(t)$$

$$\underset{(3.2.1)}{\leq} V'(t,x(t)) + L\, |p(t)|\, |u'(t)|$$

and the proof is complete.

Lemma 3.2.2.

At the points of discontinuity t_k , $k = 1,2,\ldots$ of u, the function $V(t,x)$ satisfies the estimate

$$|V(t_k,x(t_k)) - V(t_k,x(t_k^-))| \leq L\, |p(t_k)|\, |u(t_k) - u(t_k^-)|$$

where $x(t_k^-)$ denotes the left hand limit of x at t_k.

Proof.

By theorem 2.4.1 we know that $x(t)$ is a solution of (3.2.2) through (t_o,x_o) if and only if $x(t)$ satisfies the integral equation

$$x(t) = x_o + \int_{t_o}^{t} F(s,x(s))\, ds + \int_{t_o}^{t} p(s)\, du(s), \quad t \geq t_o.$$

By definition

$$|x(t_k) - x(t_k^-)| = \lim_{h \to 0^+} |x(t_k) - x(t_k-h)|$$

$$= \lim_{h \to 0^+} \left| \int_{t_k-h}^{t_k} F(s,x(s))ds + \int_{t_k-h}^{t_k} p(s)du(s) \right| \tag{3.2.3}$$

Clearly the first limit on the right hand side of (3.2.3) is zero because of continuity of F and we shall prove that

$$\lim_{h \to 0^+} \left| \int_{t_k - h}^{t_k} p(s)\, du(s) \right| = |p(t_k)\,(u(t_k) - u(t_k^-))| \tag{3.2.4}$$

Consider the positive set function μ defined by

$$\mu(A) = \left| \int_A p(s)\, du(s) \right|.$$

Let $h_1 \geq h_2 \geq h_3 \geq \cdots > 0$ and let $A_n = [t_k - h_n, t_k]$ and $h_n \to 0$ as $n \to \infty$. Then $A_1 \supset A_2 \supset A_3 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = A_0$ where $A_0 = \{t_k\}$. Therefore by theorem 1.19(e) of [53], $\mu(A_n) \to \mu(A_0)$. But $\mu(A_0) = |p(t_k)\,(u(t_k) - u(t_k^-))|$, by Ex. S, P. 199 of [38]. Thus (3.2.4) is established and from (3.2.3) we have,

$$|x(t_k) - x(t_k^-)| \leq |p(t_k)|\; |u(t_k) - u(t_k^-)|.$$

Since $V(t,x)$ is uniformly Lipschitzian in x, and this property is guaranteed because of $|\nabla V(t,x)| \leq L$, we finally have

$$\begin{aligned} |V(t_k, x(t_k)) - V(t_k, x(t_k^-))| &\leq L\, |x(t_k) - x(t_k^-)| \\ &\leq L\, |p(t_k)|\; |u(t_k) - u(t_k^-)|. \end{aligned}$$

This completes the proof.

Theorem 3.2.1.

If the trivial solution of (3.2.1) is uniformly stable, then it is also integrally stable.

Proof.

Since the trivial solution of (3.2.1) is uniformly stable, there exists a Liapunov function $V(t,x)$ on $J \times S_\rho$ satisfying the following hypotheses :

(i) $b(|x|) \leq V(t,x) \leq a(|x|)$,

(ii) $|\nabla V(t,x)| \leq L$,

(iii) $V'_{(3.2.1)}(t,x) \leq 0$,

where $a, b \in \mathcal{K}$ and L is a positive constant.

Since $x(t)$ is a solution of (3.2.2), in view of lemma 3.2.1 and hypothesis (iii), it follows that for $t \in [t_{k-1}, t_k)$, $k = 1,2,3,\ldots$

$$V'_{(3.2.2)}(t,x(t)) \leq V'_{(3.2.1)}(t,x(t)) + L\,|p(t)|\,|u'(t)|$$
$$\leq L\,|p(t)|\,|u'(t)|. \tag{3.2.5}$$

Integrating (3.2.5) for $t \in [t_{k-1}, t_k)$, we get

$$V(t,x(t)) \leq V(t_{k-1}, x(t_{k-1})) + L \int_{[t_{k-1},t]} |p(s)|\,|u'(s)|\,ds. \tag{3.2.6}$$

Since $V(t,x)$ is continuous in t for each fixed x, we have

$$V(t_k, x(t_k)) \leq |V(t_k,x(t_k)) - V(t_k,x(t_k^-))| + V(t_k,x(t_k^-))$$
$$= |V(t_k,x(t_k)) - V(t_k,x(t_k^-))| + \lim_{h \to 0^+} V(t_k-h, x(t_k-h)).$$

Using lemma 3.2.2 and (3.2.6), we obtain

$$V(t_k,x(t_k)) \leq L\,|p(t_k)|\,|u(t_k)-u(t_k^-)|$$

$$+\lim_{h\to 0^+}\Big[V(t_{k-1},x(t_{k-1}))+L\int_{[t_{k-1},t_k-h]}|p(s)|\,|u'(s)|\,ds\Big].$$

That is

$$V(t_k,x(t_k)) \leq V(t_{k-1},x(t_{k-1}))+L\,|p(t_k)|\,|u(t_k)-u(t_k^-)|$$

$$+L\int_{[t_{k-1},t_k)}|p(s)|\,|u'(s)|\,ds \qquad (3.2.7)$$

Thus the inequality (3.2.6) gives for $t \in [t_o,t_1)$

$$V(t,x(t)) \leq V(t_o,x_o)+L\int_{[t_o,t]}|p(s)|\,|u'(s)|\,ds,$$

and for $t \in [t_1,t_2)$

$$V(t,x(t)) \leq V(t_1,x(t_1))+L\int_{[t_1,t]}|p(s)|\,|u'(s)|\,ds.$$

Hence for $t \in [t_o,t_2)$, using (3.2.7) we get that

$$V(t,x(t)) \leq V(t_o,x_o)+L\,|p(t_1)|\,|u(t_1)-u(t_1^-)|$$

$$+L\int_{[t_o,t_1)}|p(s)|\,|u'(s)|\,ds+L\int_{[t_1,t]}|p(s)|\,|u'(s)|\,ds,$$

$$\leq V(t_o,x_o)+L\,|p(t_1)|\,|u(t_1)-u(t_1^-)|$$

$$+L\sum_{k=1}^{2}\int_{[t_{k-1},t_k)}|p(s)|\,|u'(s)|\,ds.$$

Therefore, in general, for $t \geq t_o$, where $t_o < t_1 < \dots < t_n = t$, we have

$$V(t,x(t)) \leq V(t_0,x_0) + L \left[\sum_{k=1}^{n-1} |p(t_k)| \; |u(t_k) - u(t_k^-)| \right.$$

$$\left. + \sum_{k=1}^{n} \int_{[t_{k-1},t_k)} |p(s)| \; |u'(s)| \, ds \right].$$

That is

$$V(t,x(t)) \leq V(t_0,x_0) + L \int_{t_0}^{t} |p(s)| \; dv_u(s). \tag{3.2.8}$$

Now let $0 < \varepsilon < \rho$ be given. Choose $\delta = \delta(\varepsilon) > 0$, $0 < \delta < \varepsilon$ such that $\int_{t_0}^{\infty} |p(s)| \; dv_u(s) < \delta$ and $a(\delta) + L\delta < b(\varepsilon)$. For $|x_0| < \delta$ and $\int_{t_0}^{\infty} |p(s)| \; dv_u(s) < \delta$, we have from (i) and (3.2.8)

$$\begin{aligned} b(|x(t)|) \leq V(t,x(t)) &\leq V(t_0,x_0) + L \int_{t_0}^{\infty} |p(s)| \; dv_u(s) \\ &\leq a(|x_0|) + L\delta \\ &< a(\delta) + L\delta < b(\varepsilon). \end{aligned}$$

This implies that $|x(t)| < \varepsilon$ whenever $|x_0| < \delta$ and $\int_{t_0}^{\infty} |p(s)| \; dv_u(s) < \delta$, for all $t \geq t_0$ and thus the proof is complete.

In the following theorem we prove the integral asymptotic stability of the trivial solution of (3.2.1).

Theorem 3.2.2.

If the trivial solution of (3.2.1) is uniformly asymptotically stable, then it is also integrally asymptotically stable.

Proof.

Since the trivial solution of (3.2.1) is uniformly asymptotically stable, by theorem 2.2.1, there exists a Liapunov function $V(t,x)$ on $J \times S_\rho$ with the following properties :

$$b(|x|) \leq V(t,x) \leq a(|x|), \tag{3.2.9}$$

$$|\nabla V(t,x)| \leq L , \tag{3.2.10}$$

and $$V'_{(3.2.1)}(t,x) \leq - c(|x|) , \tag{3.2.11}$$

where a, b, $c \in K$ and L is a positive constant.

Without loss of generality we can suppose that $L \geq 1$. Let $x(t) = x(t,t_o,x_o)$ be the unique solution of (3.2.2) existing to the right of $t_o \geq 0$. By theorem 3.2.1 the trivial solution of (3.2.1) is integrally stable. That is given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequalities $|x_o| < \delta$ and $\int_{t_o}^{\infty} |p(s)|\, dv_u(s) < \delta$ imply $|x(t)| < \varepsilon$ for all $t \geq t_o$.

Let $0 < \eta < \varepsilon$ be given. Choose $\delta_o = \delta(\rho)$, $\alpha(\eta) = \dfrac{\delta(\eta)}{2L}$

$$\text{and } T(\eta) = \frac{2a(\delta_o) + \delta(\eta)}{2c(\delta(\eta))} \tag{3.2.12}$$

It is clear that T depends only on η , not on t_o or x_o. Let $|x_o| < \delta_o$. Now we claim that there exists a $t^* \in [t_o, t_o + T(\eta)]$ such that

$$|x(t^*)| < \delta(\eta) \tag{3.2.13}$$

whenever $|x_o| < \delta_o$ and $\int_{t_o}^{\infty} |p(s)|\, dv_u(s) < \alpha(\eta)$.

Suppose not. Then $\delta \leq |x(t)| < \rho$ for all $t \in [t_o, t_o + T(\eta)]$.
Proceeding exactly on the same lines as in the proof of theorem 3.2.1 we obtain for all $t \in [t_o, t_o + T(\eta)]$

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta)\, T + L \int_{t_o}^{t_o+T} |p(s)|\, dv_u(s)$$

$$\leq a(|x_o|) - c(\delta)\, T + L\, \alpha(\eta)$$

$$\leq a(\delta_o) - c(\delta)\, T + \frac{\delta(\eta)}{2} = 0 ,$$

a contradiction, proving (3.2.13).

Thus from integral stability of (3.2.1), we have

$$|x(t)| < \eta \quad \text{for all } t \geq t^*$$

and in particular

$$|x(t)| < \eta \quad \text{for all } t \geq t_o + T(\eta)$$

whenever $|x_o| < \delta_o$ and $\int_{t_o}^{\infty} |p(s)|\, dv_u(s) < \alpha$.

This completes the proof of theorem 3.2.2.

Remark 3.2.1.

If the perturbations in (3.2.2) are not impulsive, that is, if the state of the system changes continuously with respect to time then our results reduce to some of the results of [55].

Remark 3.2.2.

Instead of $p(t)\, Du$ in (3.2.2), we can as well consider $G(t,x)\, Du$ with $|G(t,x)| \leq |p(t)|$ for sufficiently small $|x|$ and obtain the corresponding results (with minor changes) with respect to the system (2.4.1).

3.3 Uniform Asymptotic Stability under Interval Bounded Perturbations

In this section, we obtain sufficient conditions for uniform asymptotic stability of the zero solution of (3.2.1) for a larger class of perturbations namely interval bounded perturbations (see definition below). We need the following definitions in our subsequent discussion.

Definition 3.3.1.

A function $p : [0,\infty) \to R^n$ is said to be interval bounded with respect to v_u if it is measurable and $\sup_{t \geq 0} \int_t^{t+1} |p(s)| dv_u(s) < \infty$.

We will denote the space of interval bounded perturbations by B and the norm of $p \in B$ is defined by

$$||p||_B = \sup_{t \geq 0} \int_t^{t+1} |p(s)| \, dv_u(s).$$

Definition 3.3.2.

A function $h(t)$ is said to be absolutely diminishing with respect to v_u if and only if $\int_t^{t+1} |h(s)| \, dv_u(s) \to 0$ as $t \to \infty$. $x(t) = x(t, t_o, x_o)$ be any solution of (3.2.2) existing to the right of $t_o \geq 0$.

Definition 3.3.3.

The zero solution of (3.2.1) is said to be uniformly stable under B perturbations if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$, such that the inequalities $|x_o| < \delta$ and $||p||_B < \delta$ imply $|x(t)| < \varepsilon$ for all $t \geq t_o$.

Definition 3.3.4.

The zero solution of (3.2.1) is said to be uniformly attracting under B perturbations if there exists a $\delta_o > 0$ and for each $\eta > 0$ there exist $T = T(\eta) > 0$ and $\alpha = \alpha(\eta) > 0$ such that the inequalities $|x_o| < \delta_o$ and $||p||_B < \alpha$ imply $|x(t)| < \varepsilon$ for all $t \geq t_o + T$.

Definition 3.3.5.

The zero solution of (3.2.1) is said to be uniformly asymptotically stable under B perturbations if it is uniformly stable and uniformly attracting under B perturbations.

Lemma 3.3.1.

Assume that $\int_t^{t+1} |p(s)|\, dv_u(s) \leq \delta$ for all $t \geq 0$. Then, for every $t_2 > t_1$, $\int_{t_1}^{t_2} |p(s)|\, dv_u(s) < (t_2 - t_1 + 1)\,\delta$.

Proof. The proof is trivial if $t_2 - t_1$ is an integer. Therefore, let us suppose that $t_2 - t_1$ is not an integer. Let N be the integral part of $t_2 - t_1$. Then $N < t_2 - t_1 < N + 1$.

Choose $t_3 = t_1 + N + 1$. Clearly $t_3 - t_1$ is an integer and $t_3 < t_2 + 1$.

$$\text{Then } \int_{t_1}^{t_2} |p(s)|\, dv_u(s) < \int_{t_1}^{t_3} |p(s)|\, dv_u(s)$$
$$< (t_3 - t_1)\,\delta$$
$$< (t_2 - t_1 + 1)\,\delta.$$

This completes the proof.

Theorem 3.3.1.

Assume that the trivial solution of (3.2.1) is uniformly asymptotically stable. Then the trivial solution of (3.2.1) is uniformly asymptotically stable under B perturbations.

Proof.

Since the trivial solution of (3.2.1) is uniformly asymptotically stable, by theorem 2.2.1, there exists a Liapunov function $V(t,x)$ on $J \times S_\rho$ with the following properties :

$$b(|x|) \leq V(t,x) \leq a(|x|), \tag{3.3.1}$$

$$|\nabla V(t,x)| \leq L, \tag{3.3.2}$$

and

$$V'_{(3.2.1)}(t,x) \leq - c(|x|), \tag{3.3.3}$$

where a, b, $c \in K$ and L is a positive constant.

Without loss of generality we can suppose that $L \geq 1$. Let $x(t) = x(t,t_o,x_o)$ be the unique solution of (3.2.2) through (t_o,x_o) existing to the right of $t_o \geq 0$. Let $0 < \varepsilon < \rho$ be given. Choose $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$ such that

$$2a(\delta) + \delta < b(\varepsilon). \tag{3.3.4}$$

Let $\beta > 0$ be such that $\beta < \delta$ and $\beta < 2c(\beta)$.

Let $|x_o| < \delta$. We claim that the trivial solution of (3.2.1) is uniformly stable under B perturbations. Suppose not. Then there

exists an absolutely diminishing function p with respect to v_u and $T_1 > t_o$ with $||p||_B \leq \frac{\beta}{2L}$ such that

$$|x(T_1)| \geq \varepsilon. \tag{3.3.5}$$

The existence of such a T_1 is guaranteed because of the right continuity of $x(t)$.

Since the solution $x(t)$ of (3.2.2) is unique to the right of t_o, there exists a nonnegative number δ_1 such that

$$\inf_{t \in [t_o, T_1]} |x(t)| = \delta_1 .$$

If $\delta_1 = 0$ for some $t^* \in [t_o, T_1]$ then by the uniqueness of solution $x(t)$ of (3.2.2) to the right of t_o we have $x(t) \equiv 0$ for all $t \geq t^*$ and thus the proof is trivial in this case. Therefore we consider the case where $\delta_1 > 0$. Then we have $\delta_1 \leq |x(t)| < \rho$ between t_o and T_1. Let t_k, $k = 1,2,\ldots$ be the points of discontinuity of u in $[t_o, T_1]$.

In view of (3.3.3) and lemma 3.2.1, we have for $t \in [t_{k-1}, t_k)$,

$$\begin{aligned} V'_{(3.2.2)}(t,x(t)) &\leq V'_{(3.2.1)}(t,x(t)) + L\,|p(t)|\,|u'(t)| \\ &\leq -c(|x(t)|) + L\,|p(t)|\,|u'(t)| \\ &\leq -c(\delta_1) + L\,|p(t)|\,|u'(t)|. \end{aligned} \tag{3.3.6}$$

Integrating (3.3.6) with respect to t between $[t_{k-1}, t_k)$,

$$V(t,x(t)) \leq V(t_{k-1},x(t_{k-1})) - c(\delta_1)(t-t_{k-1}) + L \int_{[t_{k-1},t]} |p(s)|\,|u'(s)|\,ds. \tag{3.3.7}$$

Since $V(t,x)$ is continuous in t for each fixed x, using (3.3.7) and lemma 3.2.2, we have

$$V(t_k,x(t_k)) \leq V(t_{k-1},x(t_{k-1})) - c(\delta_1)\,(t_k - t_{k-1})$$

$$+ L\,|p(t_k)|\;|u(t_k) - u(t_k^-)|$$

$$+ L \int_{[t_{k-1},t_k)} |p(s)|\;|u'(s)|\;ds. \qquad (3.3.8)$$

From (3.3.7) we have, for $t \in [t_o,t_1)$

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta_1)(t-t_o) + L \int_{[t_o,t]} |p(s)|\;|u'(s)|\;ds,$$

and similarly for $t \in [t_1,t_2)$

$$V(t,x(t)) \leq V(t_1,x(t_1)) - c(\delta_1)\,(t-t_1) + L \int_{[t_1,t]} |p(s)|\;|u'(s)|\;ds.$$

Hence for $t \in [t_o,t_2)$, using (3.3.8) and the above estimates, we get

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta_1)(t-t_o) + L \sum_{k=1}^{2} \int_{[t_{k-1},t_k)} |p(s)|\;|u'(s)|\;ds$$

$$+ L\,|p(t_1)|\;|u(t_1) - u(t_1^-)|.$$

Thus in general for $t \in [t_o,T_1]$, where $t_o < t_1 < \dots < t_n = t$,

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta_1)(t - t_o)$$

$$+ L\,\{\sum_{k=1}^{n-1} |p(t_k)|\;|u(t_k)-u(t_k^-)| + \sum_{k=1}^{n} \int_{[t_{k-1},t_k)} |p(s)|\;|u'(s)|\;ds\}.$$

That is,

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta_1)(t-t_o) + L \int_{t_o}^{t} |p(s)|\;dv_u(s). \qquad (3.3.9)$$

Therefore at $t = T_1$,

$$V(T_1,x(T_1)) \leq V(t_o,x_o) - c(\delta_1)(T_1-t_o) + L\int_{t_o}^{T_1} |p(s)|\, dv_u(s) \quad (3.3.10)$$

Now from (3.3.4), (3.3.5), (3.3.10), lemma 3.3.1 and for $\beta = \delta_1$, we have

$$\begin{aligned} b(\varepsilon) \leq b(|x(T_1)|) \leq V(T_1,x(T_1)) &\leq V(t_o,x_o) - c(\delta_1)(T_1-t_o) + (T_1-t_o+1)\frac{\delta_1}{2} \\ &\leq a(|x_o|) - [c(\delta_1) - \frac{\delta_1}{2}]\,(T_1 - t_o) + \frac{\delta_1}{2} \\ &\leq a(\delta) + \frac{\delta_1}{2} \\ &< a(\delta) + \frac{\delta}{2}\ , \text{ since } \delta_1 < \delta, \\ &< b(\varepsilon) \end{aligned}$$

a contradiction. Hence the trivial solution of (3.2.1) is uniformly stable under B perturbations.

Let $0 < \eta < \varepsilon$ be given. For the rest of the proof

$$\text{choose } \delta_o = \delta(\rho) \text{ and } \alpha(\eta) = \frac{\delta(\eta)}{2} \text{ and } T(\eta) = \frac{6a(\delta_o) + 2\alpha(\eta)}{\alpha(\eta)} \quad (3.3.11)$$

It is clear that T depends only on η, not on t_o or x_o. We claim that there exists a $t^* \in [t_o,t_o+T(\eta)]$ such that

$$|x(t^*)| < \delta(\eta) \quad (3.3.12)$$

whenever $|x_o| < \delta_o$ and $||p||_B < \frac{\alpha}{3L}$.

Suppose not. Then $\delta \leq |x(t)| < \rho$ for all $t \in [t_o,t_o+T(\eta)]$. Proceeding on the same lines as in the first part of the proof we obtain for all $t \in [t_o,t_o + T(\eta)]$,

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta)(t-t_o) + L\int_{t_o}^{t} |p(s)|\, dv_u(s).$$

At $t = t_o + T$

$$V(t_o+T,x(t_o+T)) \leq V(t_o,x_o) - c(\delta)T + L\int_{t_o}^{t_o+T} |p(s)|\, dv_u(s).$$

Applying lemma 3.3.1 and using (3.3.11), we get

$$\begin{aligned} V(t_o+T,x(t_o+T)) &\leq V(t_o,x_o) - c(\delta)\, T + (T+1)\frac{\alpha}{3} \\ &\leq a(|x_o|) - c(\delta/2)T + T\frac{\alpha}{3} + \frac{\alpha}{3} \\ &< a(\delta_o) - c(\alpha)\, T + T\frac{\alpha}{3} + \frac{\alpha}{3}\,. \\ &< a(\delta_o) - \frac{\alpha}{2}\, T + \frac{\alpha}{3}\, T + \frac{\alpha}{3} \\ &= a(\delta_o) - \frac{\alpha}{6}\, T + \frac{\alpha}{3} = 0 \end{aligned}$$

which is a contradiction, proving (3.3.12).

Hence the uniform stability implies that

$$|x(t)| < \eta \quad \text{for all } t \geq t^*$$

and in particular

$$|x(t)| < \eta \quad \text{for all } t \geq t_o + T(\eta)$$

whenever $|x_o| < \delta_o$ and $||p||_B < \frac{\alpha}{3L}$.

This completes the proof of the theorem.

Remark 3.3.1.

Observe that a measurable function $p(\cdot)$ is interval bounded if, for example, $\int_{t_o}^{\infty} |p(s)|\, dv_u(s) < \infty$. Therefore, if $x \equiv 0$ of (3.2.1)

is uniformly asymptotically stable under B perturbations, then it is also integrally asymptotically stable. Thus theorems 3.2.1 and 3.2.2 are immediate consequences of theorem 3.3.1.

Remark 3.3.2.

In the case of continuous perturbations i.e. the state of the system changes continuously with respect to time, theorem 3.3.1 reduces to theorem 4 of [55] .

3.4 Uniform and Uniform Ultimate Boundedness

A few results on uniform and uniform ultimate boundedness properties of solutions of (3.2.2) are presented in this section, assuming the corresponding properties of solutions of (3.2.1). The results of this section basically depend on the converse theorems developed by Yoshizawa [61] for the above said properties of solutions of (3.2.1).

Definition 3.4.1.

A solution $x(t,t_o,x_o)$ of (3.2.1) is uniformly bounded, if for any $\alpha > 0$, there exists a $\beta(\alpha) > 0$ such that the inequality $|x_o| \leq \alpha$ implies that $|x(t,t_o,x_o)| < \beta$ for all $t \geq t_o$.

Definition 3.4.2.

A solution $x(t,t_o,x_o)$ of (3.2.1) is uniformly ultimately bounded for bound D, if there exists a $D > 0$ and corresponding to any $\alpha > 0$ there exists a $T = T(\alpha) > 0$ such that the inequality

$|x_0| \leq \alpha$ implies that $|x(t,t_0,x_0)| < D$ for all $t \geq t_0+T$.

Now we present the following theorems.

Theorem 3.4.1.

Let $F(t,x) \in \bar{C}_0(x)$ and the solutions of (3.2.1) are uniformly bounded. Then the solutions of (3.2.2) are also uniformly bounded, provided $\int_{t_0}^{\infty} |p(s)| \, dv_u(s) < \infty$.

Proof.

Since the solutions of (3.2.1) are uniformly bounded and $F(t,x) \in \bar{C}_0(x)$, by theorem 20.1 [61,p. 105] there exists a Liapunov function $V(t,x)$ defined on $0 \leq t < \infty$, $|x| \geq R$, where $R > 0$ may be large, satisfying

$$a(|x|) \leq V(t,x) \leq b(|x|) \tag{3.4.1}$$

where $a(r)$ and $b(r)$ are continuous increasing functions and $a(r) \to \infty$ as $r \to \infty$,

$$V(t,x) \in \bar{C}_0(x), \text{ i.e., } |\nabla V(t,x)| \leq L, \text{ where } L \tag{3.4.2}$$

is a positive constant, and

$$V'_{(3.2.1)}(t,x) \leq 0. \tag{3.4.3}$$

Now for a solution $y(t) = y(t,t_0,x_0)$ of (3.2.2) and $t \in [t_{k-1}, t_k)$, we have from lemma 3.2.1 and (3.4.3)

$$\begin{aligned} V'_{(3.2.2)}(t,y(t)) &\leq V'_{(3.2.1)}(t,y(t)) + L\,|p(t)|\,|u'(t)| \\ &\leq L\,|p(t)|\,|u'(t)|. \end{aligned}$$

After a little computation as in theorem 3.2.1, using lemma 3.2.2 and the properties of $V(t,x)$, we finally obtain, for $t \geq t_0$,

$$V(t,y(t)) \leq V(t_0,y_0) + L \int_{t_0}^{t} |p(s)|\, dv_u(s). \tag{3.4.4}$$

Let $\alpha > 0$ be given. Choose $\beta = \beta(\alpha) > 0$ such that

$$b(\alpha) + LM < a(\beta) \text{ where } \int_{t_0}^{\infty} |p(s)|\, dv_u(s) \leq M.$$

Let $|y_0| \leq \alpha$. Then, we claim that all the solutions of (3.2.2) are uniformly bounded.

Suppose not. Then there exists a $t^* \in J$ such that

$$|y(t^*)| \geq \beta$$

whenever $|y_0| \leq \alpha$. From (3.4.4) for $t = t^*$ we have

$$\begin{aligned} a(\beta) \leq a(|y(t^*)|) \leq V(t^*,y(t^*)) &\leq V(t_0,x_0) + L \int_{t_0}^{\infty} |p(s)|\, dv_u(s) \\ &\leq b(|y_0|) + LM \\ &\leq b(\alpha) + LM < a(\beta), \end{aligned}$$

a contradiction and this completes the proof.

The next theorem gives sufficient conditions for uniform ultimate boundedness of solutions of (3.2.2) for a bound D.

Theorem 3.4.2.

Let $F(t,x) \in \bar{C}_0(x)$ and the solutions of (3.2.1) be uniformly and uniformly ultimately bounded for a bound R^*. Then the solutions of (3.2.2) are uniformly ultimately bounded for some bound D, provided

$$\int_{t_0}^{\infty} |p(s)|\, dv_u(s) < \infty.$$

Proof.

Since the solutions of (3.2.1) are uniformly and uniformly ultimately bounded for a bound R^*, in view of theorem 20.4 [61, p. 107] there exists a Liapunov function $V(t,x)$ defined on $0 \leq t < \infty$ and $|x| \geq R$, $0 < R^* < R$, with the following properties :

$$a(|x|) \leq V(t,x) \leq b(|x|) \tag{3.4.5}$$

where $a(r)$ and $b(r)$ are continuous increasing functions, $a(r) > 0$ for $r \geq R^*$ and $a(r) \to \infty$ as $r \to \infty$,

$$V'_{(3.2.1)}(t,x) \leq - c\ V(t,x) \tag{3.4.6}$$

where c is a positive constant,

and $$V \in \bar{C}_0(x) \ . \tag{3.4.7}$$

From the relations (3.4.5), (3.4.6) and (3.4.7) it is clear that

$$|\nabla V(t,x)| \leq L \text{ and } V'_{(3.2.1)}(t,x) \leq - c\ a(R).$$

Let $y(t) = y(t,t_0,x_0)$ be any solution of (3.2.2) existing to the right of $t_0 \geq 0$. Lemma 3.2.1 yields, for $t \in [t_{k-1},t_k)$, that

$$\begin{aligned} V'_{(3.2.2)}(t,y(t)) &\leq V'_{(3.2.1)}(t,y(t)) + L\ |p(t)|\ |u'(t)|, \\ &\leq - c\ a(R) + L\ |p(t)|\ |u'(t)|. \end{aligned}$$

Proceeding as in theorem 3.2.1, we obtain

$$V(t,y(t)) \leq V(t_0,x_0) - c\ a(R)\ (t - t_0) + L \int_{t_0}^{t} |p(s)|\ dv_u(s). \tag{3.4.8}$$

Let $\alpha > 0$ be given. Choose D such that $b(\alpha) + LM < 2a(D)$ where $\int_{t_0}^{\infty} |p(s)| \, dv_u(s) \leq M$. Select $T(\alpha) = \dfrac{b(\alpha) + LM}{2c \, a(R)}$. Clearly T depends only on α.

We claim that all the solutions of (3.2.2) are uniformly ultimately bounded for the bound D. That is, for $D > 0$ and corresponding to any $\alpha > 0$ there exists a $T = T(\alpha) > 0$ such that $|x_0| \leq \alpha$ implies $|y(t)| < D$ for all $t \geq t_0 + T$.

Suppose not. Then there exists a $t^* \geq t_0 + T$, such that

$$|y(t^*)| \geq D, \tag{3.4.9}$$

whenever $|x_0| \leq \alpha$. Therefore from (3.4.8) for $t = t^*$, we have

$$\begin{aligned} a(D) \leq a(|y(t^*)|) &\leq V(t^*, y(t^*)) \\ &\leq V(t_0, x_0) - c \, a(R) \, (t^* - t_0) + L \int_{t_0}^{\infty} |p(s)| \, dv_u(s) \\ &\leq b(|x_0|) - c \, a(R) \, T + LM \\ &\leq b(\alpha) - c \, a(R) T + LM = \frac{b(\alpha) + LM}{2} < a(D), \end{aligned}$$

a contradiction to (3.4.9). Hence our claim is true. Therefore all the solutions of (3.2.2) are uniformly ultimately bounded for a bound D. This completes the proof.

Remark 3.4.1.

Suppose the perturbations are not impulsive; that is, the state of system changes continuously with respect to time and if the solutions of (3.2.1) are uniformly and uniformly ultimately bounded then they are totally bounded [cf. 61, p. 120].

3.5 Eventual Uniform Asymptotic Stability

The purpose of this section is to give new results on the preservation (actually, the "eventualization") of uniform asymptotic stability from the system (3.2.1) to the perturbed system

$$Dx = F(t,x) + G(t,x)\, Du \tag{3.5.1}$$

where $F : J \times R^n \to R^n$ is continuous under a general class of perturbations.

Definition 3.5.1.

The trivial solution $x \equiv 0$ of (3.2.1) is said to be eventually uniformly asymptotically stable if the following two conditions hold :

(i) for every $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$ and $\tau = \tau(\varepsilon) > 0$ such that $|x(t,t_o,x_o)| < \varepsilon$, $t \geq t_o \geq \tau(\varepsilon)$, provided $|x_o| < \delta$,

(ii) for every $\eta > 0$, there exist positive numbers δ_o, τ_o and $T = T(\eta)$ such that $|x(t,t_o,x_o)| < \eta$, $t \geq t_o+T$, $t_o \geq \tau_o$, provided $|x_o| < \delta_o$.

Now we prove the following main theorem of this section.

Theorem 3.5.1.

Assume that

(H_1) $F \in \bar{C}_o(x)$, $F(t,0) = 0$ for $t \geq 0$ and the trivial solution of (3.2.1) is eventually uniformly asymptotically stable.

(H_2) there exists $r > 0$ such that for every λ , $0 < \lambda < r$, there exist $\tau_\lambda \geq 0$ and a measurable function $p_\lambda(t)$ defined on $[\tau_\lambda, \infty)$ such that $|G(t,x)| \leq p_\lambda(t)$ for all $\lambda \leq |x| < r$ and $t \geq \tau_\lambda$, where

$$G_\lambda(t) = \int_t^{t+1} p_\lambda(s)\, dv_u(s) \to 0 \text{ as } t \to \infty .$$

Define

$$Q_\lambda(t) = \sup \{G_\lambda(s) : t - 1 \leq s < \infty\}.$$

Obviously, $Q_\lambda(t) \searrow 0$ as $t \to \infty$.

Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$ and $|x_0| < \delta_0$, then the solution $x(t) = x(t,t_0,x_0)$ of (3.5.1) satisfies $|x(t)| \to 0$ as $t \to \infty$. In particular, if $G(t,0) \equiv 0$, then the trivial solution of (3.5.1) is eventually uniformly asymptotically stable.

Proof.

Since the solutions of (3.2.1) are unique and continuous with respect to initial values, in view of remark 3.14.1 in [23], the trivial solution of (3.2.1) is uniformly asymptotically stable. Thus, by theorem 2.2.1, there exists a Liapunov function $V(t,x)$ on $J \times S_r$ satisfying

(i) $b(|x|) \leq V(t,x) \leq a(|x|)$,

(ii) $V'_{(3.2.1)}(t,x) \leq - c(|x|)$,

(iii) $|\nabla V(t,x)| \leq L$,

where a, b, $c \in K$ and $L > 0$ is a constant.

Let $0 < \varepsilon < r$ be given. Choose $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$ so that

$$2\, a(\delta) < b(\varepsilon), \tag{3.5.2}$$

Let $\alpha > 0$ be any number such that $\alpha < \delta$. Choose $T_1 = T_1(\varepsilon) \geq \tau_\delta + 1$

so that

$$2L\, Q_{\alpha}(T_1) < \min\, [c(\alpha),\, b(\varepsilon)]\, . \tag{3.5.3}$$

Let $|x_0| < \delta$ and $t_0 \geq T_1$. Then we claim that

$$|x(t,t_0,x_0)| < \varepsilon \quad \text{for } t \in [t_0,\infty). \tag{3.5.4}$$

Suppose not. Let T_3 be the first point larger than t_0 such that $|x(T_3)| \geq \varepsilon$; its existence is guaranteed by the right continuity of $x(t)$.

Since the solution $x(t)$ of (3.5.1) is unique to the right of t_0, there exists a nonnegative number δ_1 such that

$$\inf_{t \in [t_0,T_3]} |x(t)| = \delta_1.$$

If $\delta_1 = 0$ for some $t^* \in [t_0,T_3]$ then by the uniqueness of solution $x(t)$ of (3.5.1) to the right of t_0 we have $x(t) \equiv 0$ for all $t \geq t^*$ and thus the proof is trivial in this case. Therefore we consider the case where $\delta_1 > 0$. We now have $\delta_1 \leq |x(t)| < r$ between t_0 and T_3. Without loss of generality we can suppose that $\{t_k\}$ are the discontinuities of u in $[t_0,T_3]$. Since u is differentiable on $[t_{k-1},t_k)$, $k = 1,2,\ldots$, x is also differentiable on $[t_{k-1},t_k)$, $k = 1,2,\ldots$.

From lemma 3.2.1, we have for $t \in [t_{k-1},t_k)$,

$$V'_{(3.5.1)}(t,x(t)) \leq V'_{(3.2.1)}(t,x(t)) + L\, |G(t,x(t))|\, |u'(t)|.$$

Integrating for $t \in [t_{k-1},t_k)$ and using (ii) and (iii), we get

$$V(t,x(t)) \leq V(t_{k-1},x(t_{k-1})) - \int_{[t_{k-1},t]} c(|x(s)|)ds + L\int_{[t_{k-1},t]} |G(s,x(s))|\,|u'(s)|ds.$$

That is

$$V(t,x(t)) \leq V(t_{k-1},x(t_{k-1})) - c(\delta_1)(t-t_{k-1}) + L\int_{[t_{k-1},t]} p_{\delta_1}(s)|u'(s)|\,ds. \tag{3.5.5}$$

Since $V(t,x)$ is continuous in t for each fixed x,

$$V(t_k,x(t_k)) \leq |V(t_k,x(t_k)) - V(t_k,x(t_k^-))| + \lim_{h \to 0^+} V(t_k-h,x(t_k-h)).$$

Using (3.5.5) and lemma 3.2.2, we get

$$\begin{aligned} V(t_k,x(t_k)) \leq\ & L\, p_{\delta_1}(t_k)\ |u(t_k) - u(t_k^-)| \\ & + \lim_{h \to 0^+} [V(t_{k-1},x(t_{k-1})) - c(\delta_1)(t_k-t_{k-1}-h) \\ & + L\int_{[t_{k-1},t_k-h]} p_{\delta_1}(s)\ |u'(s)|\ ds]. \end{aligned}$$

That is

$$\begin{aligned} V(t_k,x(t_k)) \leq\ & V(t_{k-1},x(t_{k-1})) - c(\delta_1)(t_k-t_{k-1}) + L\, p_{\delta_1}(t_k)\ |u(t_k) - u(t_k^-)| \\ & + L\int_{[t_{k-1},t_k)} p_{\delta_1}(s)\ |u'(s)|\ ds. \end{aligned} \tag{3.5.6}$$

Therefore, we have from (3.5.5), for $t \in [t_o,t_1)$,

$$V(t,x(t)) \leq V(t_o,x_o) - c(\delta_1)(t-t_o) + L\int_{[t_o,t]} p_{\delta_1}(s)\ |u'(s)|\ ds,$$

and similarly for $t \in [t_1, t_2)$ it follows that

$$V(t,x(t)) \leq V(t_1,x(t_1)) - c(\delta_1)(t-t_1) + L \int_{[t_1,t]} p_{\delta_1}(s)\,|u'(s)|\,ds.$$

Hence for $t \in [t_0, t_2)$, by (3.5.6), we get

$$V(t,x(t)) \leq V(t_0,x_0) - c(\delta_1)(t_1-t_0) + L\, p_{\delta_1}(t_1)\,|u(t_1) - u(t_1^-)|$$

$$+ L \int_{[t_0,t_1)} p_{\delta_1}(s)|u'(s)|ds - c(\delta_1)(t-t_1) + L \int_{[t_1,t]} p_{\delta_1}(s)|u'(s)|\,ds$$

$$\leq V(t_0,x_0) - c(\delta_1)(t-t_0) + L\, p_{\delta_1}(t_1)\,|u(t_1) - u(t_1^-)|$$

$$+ L \sum_{k=1}^{2} \int_{[t_{k-1},t_k)} p_{\delta_1}(s)\,|u'(s)|\,ds.$$

In general, for $t \geq t_0$, where $t_0 < t_1 < \dots < t_n = t$,

$$V(t,x(t)) \leq V(t_0,x_0) - c(\delta_1)(t-t_0) + L\,[\sum_{k=1}^{n-1} p_{\delta_1}(t_k)\,|u(t_k) - u(t_k^-)|$$

$$+ \sum_{k=1}^{n} \int_{[t_{k-1},t_k)} p_{\delta_1}(s)\,|u'(s)|\,ds\,].$$

That is for all $t \geq t_0$,

$$V(t,x(t)) \leq V(t_0,x_0) - c(\delta_1)(t-t_0) + L \int_{t_0}^{t} p_{\delta_1}(s)\,dv_u(s).$$

Since $t \geq t_0 \geq 1$, by changing the order of integration, we obtain

$$V(t,x(t)) \leq V(t_0,x_0) - c(\delta_1)(t-t_0) + L \int_{t_0-1}^{t} G_{\delta_1}(s)\,dv_u(s)$$

$$\leq V(t_0,x_0) - c(\delta_1)\,(t-t_0) + L\,Q_{\delta_1}(t_0)(t-t_0+1).$$

That is

$$V(t,x(t)) \leq V(t_o,x_o) + [LQ_{\delta_1}(t_o)-c(\delta_1)]\,(t-t_o) + L\,Q_{\delta_1}(t_o) \qquad (3.5.7)$$

Thus at $t = T_3$, from (3.5.7) and with $\alpha = \delta_1$ in (3.5.3), we have

$$\begin{aligned} b(\varepsilon) \leq b(|x(T_3)|) \leq V(T_3,x(T_3)) \leq V(t_o,x_o) &+ [L\,Q_{\delta_1}(t_o) - c(\delta_1)]\,(T_3-t_o) \\ &+ L\,Q_{\delta_1}(t_o) \\ \leq a(|x_o|) &+ [L\,Q_{\delta_1}(T_1)-c(\delta_1)]\,(T_3-t_o) \\ &+ L\,Q_{\delta_1}(T_1) \\ \leq a(\delta) &+ L\,Q_{\delta_1}(T_1) \\ < \frac{b(\varepsilon)}{2} &+ \frac{b(\varepsilon)}{2} = b(\varepsilon), \end{aligned}$$

a contradiction, proving (3.5.4). This proves that the trivial solution of (3.5.1) is eventually uniformly stable for the case $G(t,0) \equiv 0$. For the rest of the proof fix $r_1 < r$ and choose $\delta_o = \delta(r_1)$ and $T_o = T_1(r_1)$. Fix $t_o \geq T_o$ and $|x_o| < \delta_o$. Then (3.5.4) implies that $|x(t,t_o,x_o)| < r_1$ on $[t_o,\infty)$.

Let $0 < \eta < r_1$. Choose $\delta = \delta(\eta)$, $0 < \delta < \eta$ so that $2a(\delta) < b(\eta)$ and $T_1(\eta) \geq \tau_\delta + 1$ so large that

$$2L\,Q_\delta(T_1) < \min\ [c(\delta),\, b(\eta)]\ .$$

Select $T = \dfrac{c(\delta)\ T_1(\eta) + b(\eta) + 2a(r_1)}{c(\delta)} > T_1(\eta)$.

Clearly T depends only on η, not on t_0 or x_0. We claim that

$$|x(t_1,t_0,x_0)| < \delta(\eta) \tag{3.5.8}$$

for some t_1 in $[t_0+T_1,t_0+T]$.

Suppose not. Then $\delta \le |x(t,t_0,x_0)| < r_1$ on $[t_0+T_1,t_0+T]$. Let $y_0 = x\,(t_0+T_1,t_0,x_0)$. Then by (3.5.7)

$$\begin{aligned}
0 < b(\delta) &\le b(|x(t_0+T)|) \le V(t_0+T,\ x(t_0+T)) \\
&\le a(|x(t_0+T_1)|) + [L\,Q_\delta(t_0+T_1) - c(\delta)]\,(T-T_1) \\
&\qquad + L\,Q_\delta(t_0+T_1) \\
&\le a(|y_0|) - \tfrac{1}{2}\,c(\delta)\,(T-T_1) + L\,Q_\delta(T_1) \\
&< a(r_1) - \tfrac{1}{2}\,c(\delta)\,(T-T_1) + \frac{b(\eta)}{2} = 0,
\end{aligned}$$

which is a contradiction and thus (3.5.8) is proved.

By (3.5.4)

$$|x(t,t_1,x(t_1,t_0,x_0))| < \eta \qquad \text{on } [t_1,\infty)$$

because $t_1 \ge t_0 + T_1 \ge T_1$ and $|x(t_1,t_0,x_0)| < \delta$. Hence, by uniqueness of solutions of (3.5.1), we have

$$|x(t,t_0,x_0)| < \eta \quad \text{for } t \ge t_0 + T.$$

Since η is arbitrary $|x(t,t_0,x_0)| \to 0$ as $t \to \infty$. Since T depends only on η and δ depends only on ε, the trivial solution of (3.5.1) is eventually uniformly asymptotically stable if $G(t,0) \equiv 0$. This completes the proof.

Remark 3.5.1.

A theorem on quasi-equi-asymptotic stability of the trivial solution of (3.5.1) is proved in [14, theorem 2, p. 150], in which a stronger type of stability (i.e., exponential asymptotic stability) on the trivial solution of (3.2.1) is assumed. Further, it is assumed that the jumps of u at the discontinuities t_k, $k = 1,2,\ldots$ are bounded by $\alpha e^{-c(t_k-t_o)}$, α, $c > 0$, $t_k \geq t_o$ (see [14], p. 151) so that the jumps ultimately die down to zero. In spite of all these assumptions, they could only get a very weaker type of stability for (3.5.1) i.e., quasi-equi-asymptotic stability. This has been improved in [45] and eventual uniform asymptotic stability for (3.5.1) has been obtained, assuming the same type of stability for (3.2.1). But the conditions (H_1), (H_3) and (H_4) in [45] are some what restrictive. Notice that our condition (H_2) is less restrictive, more compact and obviously covers a larger class of perturbation functions.

Remark 3.5.2.

If the perturbation functions in equation (3.5.1) are not impulsive, and the state of the system changes continuously with respect to time, then our results reduce to some of the results of [23] and [56].

Example 3.5.1.

Define a scalar function $G(t,x)$ for $0 < t < \infty$, $0 \leq |x| < r$ as follows :

$$G(t,x) = \begin{cases} 1 & \text{for } t = 2n+1,\ n \text{ is a positive integer} \\ \dfrac{1}{t(tx+1)} & \text{for other values of } t. \end{cases}$$

Let $\tau_{\delta_1} = \delta_1^{-1} + 1$. Then

$$p_{\delta_1}(2n+1) = 1 \text{ and } p_{\delta_1}(t) = \frac{1}{t(t\delta_1 - 1)} \text{ otherwise.}$$

Choose

$$u(t) = \begin{cases} t^2 & \text{for } 0 \le t < 2 \\ t^2 + \sum\limits_{k=1}^{n} 2k, & \text{for } 2n \le t < 2(n+1) \\ & n = 1,2,\ldots \ . \end{cases}$$

Then $p_{\delta_1}(t) \not\to 0$ as $t \to \infty$ and

$$\int_{\tau_{\delta_1}}^{2n} p_{\delta_1}(s)\, dv_u(s) = 2\int_{\tau_{\delta_1}}^{2n} \frac{ds}{s\delta_1 - 1} + \sum_{k=\frac{n_o}{2}}^{n-1} \frac{1}{2k\delta_1 - 1}$$

where n_o is the smallest even integer greater than τ_{δ_1}.

Clearly $\int_{\tau_{\delta_1}}^{\infty} p_{\delta_1}(s)\, dv_u(s) = \infty$.

However, for $2n \le t \le 2n+1$, we have

$$\int_{2n}^{2n+1} p_{\delta_1}(s)\, dv_u(s) = 2\int_{2n}^{2n+1} \frac{ds}{s\delta_1 - 1} + \sum_{k=\frac{n_o}{2}}^{n} \frac{1}{2k\delta_1 - 1}$$
$$- \sum_{k=\frac{n_o}{2}}^{n-1} \frac{1}{2k\delta_1 - 1}$$
$$= \frac{2}{\delta_1} \log\left(1 + \frac{\delta_1}{2n\delta_1 - 1}\right) + \frac{1}{2n\delta_1 - 1}$$

$\to 0$ as $n \to \infty$

Thus we see that the condition (H_2) of theorem 3.5.1 holds. On the other hand, the conditions (H_3) and (H_4) of [45] fail to satisfy.

3.6 Asymptotic Equivalence

Nonlinear ordinary differential systems are the best basic models for perturbation problems, because most useful properties of solutions of unperturbed systems are readily available in the literature. In this section, we consider the perturbed differential system (3.5.1) and obtain an asymptotic equivalence type correspondence between the bounded solutions of (3.2.1) and (3.5.1). Here the class of perturbations are integrable with respect to the total variation function v_u. The major advantage of our study is that, not only the system (3.2.1) is assumed to be nonlinear but also, no differentiability properties are required on F.

Definition 3.6.1.

The differential systems (3.5.1) and (3.2.1) are said to be asymptotically equivalent if, for every solution $x(t)$ of (3.5.1) ($y(t)$ of (3.2.1)), there is a solution $y(t)$ of (3.2.1) ($x(t)$ of (3.5.1)) such that $|x(t) - y(t)| \to 0$ as $t \to \infty$.

We need the following conditions for our subsequent discussion :

(C_1) For $(t,x,y) \in J \times R^n \times R^n$ and for α, $h > 0$

$$|x - y + h\,[F(t,x) - F(t,y)]| \leq (1 - \alpha h)\,|x-y| \,.$$

(C_2) If $(t,x) \in J \times R^n$, $|G(t,x)| \leq \omega(t,|x|)$

where $\omega : J \times J \to J$, $\omega(r,s) \leq \omega(r,t)$ for $s \leq t$ and

$\int_0^\infty \omega(s,c)\, dv_u(s) < \infty$ where c is a positive constant.

Throughout this section we assume the existence of bounded solutions of (3.2.1) and (3.5.1) for all $t \in J$.

Theorem 3.6.1.

Assume that the conditions (C_1) and (C_2) hold. If x and y are bounded solutions of (3.5.1) and (3.2.1) on J respectively, then $\lim_{t \to \infty} |x(t) - y(t)| = 0$.

Proof.

Let M be a positive number such that $|y(t)| \leq M$ for all $t \in J$ and let $x(t)$ be a bounded solution of (3.5.1) with a bound N such that $x(0) = y(0)$. Now (C_2) implies that $\int_0^\infty \omega(s,N)\, dv_u(s)$ is finite and thus $\lim_{t \to \infty} \int_t^\infty \omega(s,N)\, dv_u(s) = 0$.

Let L be a positive number such that $\int_0^\infty \omega(s,N)\, dv_u(s) \leq L$.

Define a function $m(t)$ on J by the relation

$$m(t) = |x(t) - y(t)| \; ;$$

clearly $m(0) = 0$. Since u is differentiable on $[t_{k-1}, t_k)$, $k = 1,2,\ldots$ x is also differentiable on $[t_{k-1}, t_k)$, $k = 1,2,\ldots$. Let $h > 0$ be sufficiently small and let $t \in [t_{k-1}, t_k)$. Now by definition, for $t \in [t_{k-1}, t_k)$,

$$m'(t) = \lim_{h \to 0^+} \frac{m(t+h) - m(t)}{h} .$$

$$= \lim_{h \to 0^+} \frac{|x(t+h) - y(t+h)| - m(t)}{h} .$$

$$= \lim_{h \to 0^+} \frac{|x(t)+hF(t,x(t))+hG(t,x(t))u'(t) - y(t)-hF(t,y(t))| - m(t)}{h} .$$

Using (C_1) and (C_2)

$$m'(t) \leq \lim_{h \to 0^+} \frac{(1 - \alpha h)\, m(t) - m(t)}{h} + |G(t,x(t))|\ |u'(t)|$$

$$\leq -\alpha\, m(t) + \omega(t,N)\ |u'(t)| . \tag{3.6.1}$$

Integrating (3.6.1) between t_{k-1} and t_k,

$$m(t) \leq m(t_{k-1})\, e^{-\alpha(t-t_{k-1})} + e^{-\alpha t} \int_{[t_{k-1},t]} e^{\alpha s}\, \omega(s,N)\ |u'(s)|\ ds. \tag{3.6.2}$$

At the discontinuous points t_k, $k = 1,2,3,\ldots$, we have

$$m(t_k) \leq |m(t_k) - m(t_k^-)| + \lim_{h \to 0^+} m(t_k - h) . \tag{3.6.3}$$

But

$$|m(t_k) - m(t_k^-)| = \big|\, |x(t_k) - y(t_k)| - |x(t_k^-) - y(t_k^-)| \,\big|$$

$$\leq |x(t_k) - x(t_k^-)| \quad \text{since } y(t_k) = y(t_k^-).$$

From (3.5.1)

$$|x(t_k) - x(t_k^-)| = \lim_{h \to 0^+} \left| \int_{t_k-h}^{t_k} F(s,x(s))ds + \int_{t_k-h}^{t_k} G(s,x(s))\, du(s) \right| .$$

From lemma 3.2.2, it is clear that the first limit on the right is zero and the second limit is $\leq |G(t_k, x(t_k))|\ |u(t_k) - u(t_k^-)|$. Therefore

$$|x(t_k) - x(t_k^-)| \leq |G(t_k, x(t_k))|\ |u(t_k) - u(t_k^-)|$$
$$\leq \omega(t_k, |x(t_k)|)\ |u(t_k) - u(t_k^-)|$$
$$\leq \omega(t_k, N)\ |u(t_k) - u(t_k^-)|\ .$$

Thus, from (3.6.2) and (3.6.3)

$$m(t_k) \leq m(t_{k-1})\, e^{-\alpha(t_k - t_{k-1})}$$
$$+ \omega(t_k, N)\ |u(t_k) - u(t_k^-)|$$
$$+ e^{-\alpha t_k} \int_{[t_{k-1}, t_k)} e^{\alpha s}\, \omega(s, N)\ |u'(s)|\ ds. \tag{3.6.4}$$

A similar computation as in theorem 3.2.1 together with the estimates (3.6.2) and (3.6.4) yield that, for $t \geq 0$

$$m(t) \leq m(0)\, e^{-\alpha t} + e^{-\alpha t} \int_0^t e^{\alpha s}\, \omega(s, N)\ dv_u(s).$$

Since $m(0) = 0$, we have

$$|x(t) - y(t)| = m(t) \leq e^{-\alpha t} \int_0^t e^{\alpha s}\, \omega(s, N)\ dv_u(s)$$
$$= e^{-\alpha t} \int_0^{t/2} e^{\alpha s}\, \omega(s, N)\ dv_u(s)$$
$$+ e^{-\alpha t} \int_{t/2}^t e^{\alpha s}\, \omega(s, N)\ dv_u(s).$$

$$\leq e^{-\alpha t} e^{\alpha t/2} \int_0^{t/2} \omega(s,N)\, dv_u(s)$$

$$+ \int_{t/2}^{t} \omega(s,N)\, dv_u(s)$$

$$< e^{\frac{-\alpha t}{2}} \int_0^{\infty} \omega(s,N)\, dv_u(s)$$

$$+ \int_{t/2}^{\infty} \omega(s,N)\, dv_u(s)$$

$$\leq L\, e^{-\alpha t/2} + \int_{t/2}^{\infty} \omega(s,N)\, dv_u(s)$$

$$\to 0 \text{ as } t \to \infty.$$

This completes the proof of the theorem.

Corollary 3.6.1.

Let (C_1) and (C_2) hold. If x_1 and x_2 are bounded solutions of (3.5.1) on J such that $x_1(0) = x_2(0)$ then

$$\lim_{t \to \infty} |x_1(t) - x_2(t)| = 0.$$

Proof. Let $y(t)$ be a bounded solution of (3.2.1) on J such that $x_1(0) = y(0) = x_2(0)$.

Then

$$|x_1(t) - x_2(t)| \leq |x_1(t) - y(t)| + |y(t) - x_2(t)| \text{ , and}$$

hence the application of theorem 3.6.1 yields the desired result.

Remark 3.6.1.

Notice that the corollary 3.6.1 establishes the asymptotic uniqueness property of solutions of the system (3.5.1). More

specifically, there may exist more than one bounded solution of (3.5.1) through the same point but the difference between any two bounded solutions tends to zero as $t \to \infty$.

Remark 3.6.2.

If the perturbations in (3.5.1) are not impulsive, then our results reduce to some of the results of [31] .

Example 3.6.1 (when $n = 1$).

Consider the following differential equations

$$y' = -e^{-t}(y-1)^2, \; y(0) = 2 \tag{3.6.5}$$

and
$$Dx = -e^{-t}(x-1)^2 + G(t,x)\, Du, \; x(0) = 2 \tag{3.6.6}$$

where
$$G(t,x) = te^{-2t}(x-1) + \frac{te^{-2t}}{2-e^{-t}} + (1-t)\, e^{-t}$$

and
$$u(t) = \begin{cases} 0 & \text{for } 0 \le t < 1 \\ t & \text{for } 1 \le t < \infty. \end{cases}$$

Here
$$\omega(t,c) = (c+3)\, te^{-t} + e^{-t} = (ct + 3t + 1)\, e^{-t}.$$

Thus we see that the condition (C_2) of the theorem 3.6.1 is satisfied. Furthermore, the solutions $y(t)$ and $x(t)$ of (3.6.5) and (3.6.6) are given by

$$y(t) = 1 + \frac{1}{2 - e^{-t}}$$

and
$$x(t) = \begin{cases} 1 + \dfrac{1}{2 - e^{-t}} & \text{for } 0 \leq t < 1 \\ 1 + \dfrac{1}{2 - e^{-t}} + t\,e^{-t} & \text{for } 1 \leq t < \infty. \end{cases}$$

Note that $x(t)$ and $y(t) \not\to 0$ as $t \to \infty$ but $|x(t) - y(t)| \to 0$ as $t \to \infty$ and therefore the equations (3.6.5) , (3.6.6) are asymptotically equivalent.

CHAPTER - 4

INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

4.1 Introduction

Functional differential equations [4,20] , Volterra integral equations [12,36] , integro-differential equations of Volterra type [24,46] and functional differential equations of Volterra type (FDEVT) [16,17,52] have been studied by various authors and many interesting results have been accumulated. Quite recently, Seifert [16,17] investigated sufficient conditions for stability, boundedness and asymptotic stability of solutions of FDEVT, when the interval of delay becomes unbounded as $t \to +\infty$, by choosing an appropriate minimal subset of a suitable space of continuous functions (Liapunov - Razumikhin conditions) along which, the derivative of the Liapunov function admits a convenient estimate.

In the first part of this chapter we exploit further Liapunov-Razumikhin conditions and obtain sufficient conditions for uniform and exponential asymptotic stability of the zero state of FDEVT, with infinite delay, by treating them as perturbed ordinary differential systems with perturbations involving delays. In the latter part, a more general type of Volterra integro-differential system containing measures is considered and sufficient conditions for eventual uniform asymptotic stability of the zero state of such systems are investigated. Applications and examples are given to illustrate the fruitfulness of the results.

Definition 4.2.1.

By a solution of (4.2.2) we mean a continuously differentiable function $x(s)$ on $t_o \le s < T \le \infty$ such that (4.2.2) is satisfied on $t_o \le t < T$ for $x_t(\cdot)$ a segment of $x(s)$.

The equation (4.2.2) may be regarded as a perturbed system of the ordinary differential system

$$x'(t) = F(t,x(t)) \tag{4.2.3}$$

We denote by $x(t,x_o)$ any solution of (4.2.2) such that $x(t_o,x_o) = x_o$, existing to the right of $t_o \ge 0$ in S_ρ.

Definition 4.2.2.

We define a set E for all $x \in R^n$ by the relation

$$E = \{x : J \to R^n : x \text{ is continuous and}$$

$$V(s,x(s)) \le \psi(V(t,x(t))), t_1 \le s \le t,\ t_1 \ge t_o\}$$

where $\psi(r)$ is continuous on $[0,\infty)$, increasing in r and $\psi(r) > r$ for $r > 0$. The conditions on the set E are generally referred to as Liapunov-Razumikhin conditions. Further we define

$$|x_t(\cdot)| = \sup_{t_o \le s \le t} |x(s)|.$$

Definition 4.2.3.

The trivial solution $x \equiv 0$ of (4.2.3) is said to be exponentially asymptotically stable, if there exists a $\alpha > 0$ and, given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|x_o| < \delta$,

$$|x(t,x_0)| \leq \varepsilon\, e^{-\alpha(t - t_0)} \quad \text{for all } t \geq t_0.$$

4.3 Functional Differential Equations with Infinite Delay

In this section we investigate sufficient conditions for uniform asymptotic stability and exponential asymptotic stability of the trivial solution of (4.2.2) assuming the existence, uniqueness and continuous dependence of solutions of (4.2.2) on the initial values.

Theorem 4.3.1.

Let $F \in \bar{C}_0(x)$ and $F(t,0) = 0$ for all $t \geq 0$. Assume that

(i) the trivial solution of (4.2.3) is uniformly asymptotically stable,

(ii) $H(t,x_t(\cdot))$ satisfies the following hypothesis : there exists $r > 0$ such that for every λ, $0 < \lambda < r$, there exists $\tau_\lambda \geq 0$ and a function $g_\lambda(t)$ continuous on $[\tau_\lambda, \infty)$ such that $|H(t,x_t(\cdot))| \leq g_\lambda(t)\, |x_t(\cdot)|$ for all $\lambda \leq |x| < r$ and $t \geq \tau_\lambda$, where

$$G_\lambda(t) = \int_t^{t+1} g_\lambda(s)\, ds \to 0 \text{ as } t \to \infty.$$

Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$, $|x_0| < \delta_0$ and $x \in E$, then the solution $x(t,x_0)$ of (4.2.2) satisfies $|x(t,x_0)| \to 0$ as $t \to \infty$. In particular if $H(t,0) = 0$ for all $t \geq t_0$, then the trivial solution of (4.2.2) is uniformly asymptotically stable.

Proof.

Since the trivial solution of (4.2.3) is uniformly asymptotically stable, by theorem 2.2.1 there exists a Liapunov function V for (4.2.3) on $J \times S_r$ satisfying

$$b(|x|) \leq V(t,x) \leq a(|x|), \tag{4.3.1}$$

$$V'_{(4.2.3)}(t,x) \leq - c(|x|) , \tag{4.3.2}$$

and

$$|\nabla V(t,x)| \leq M, \tag{4.3.3}$$

where a, b, c $\in$ K and M is a positive constant.

Now for as long as a solution $x(t) = x(t,x_0)$ of (4.2.2) exists,

$$V'(t,x(t)) = \frac{\partial}{\partial t} V(t,x(t)) + \nabla V(t,x(t)).[F(t,x(t)) + H(t,x_t(\cdot))]$$

$$= V'_{(4.2.3)}(t,x(t)) + \nabla V(t,x(t)) \, . \, H(t,x_t(\cdot))$$

$$\leq - c(|x(t)|) + M \, |H(t,x_t(\cdot))|.$$

Thus if $0 < \lambda \leq |x(t)| < r$, we have

$$V'(t,x(t)) \leq - c(|x(t)|) + M \, g_\lambda(t) \, |x_t(\cdot)|. \tag{4.3.4}$$

Therefore from (4.3.1), (4.3.4) and for x $\in$ E, we obtain

$$V'(t,x(t)) \leq - c(|x(t)|) + M \, g_\lambda(t) \, b^{-1} \, [\psi(a(|x(t)|))].$$

Integrating from t_0 to t we have

$$V(t,x(t)) \leq V(t_0,x_0) - \int_{t_0}^{t} c(|x(s)|)ds + M \int_{t_0}^{t} g_\lambda(s) \, b^{-1} \, [\psi(a(|x(s)|))] \, ds.$$

Thus if $0 < \lambda \leq |x(s)| < r$ between t_0 and t and from (4.3.1), we get

$$V(t,x(t)) \leq a(|x_0|) - c(\lambda) \, (t-t_0) + M \, b^{-1} \, [\psi(a(r))] \int_{t_0}^{t} g_\lambda(s) \, ds.$$

If $t \geq t_0 \geq 1$, then lemma 3.4 in [56] yields

$$V(t,x(t)) \leq a(|x_o|) - c(\lambda)(t-t_o) + M b^{-1} [\psi(a(r))] \int_{t_o-1}^{t} G_\lambda(s)\, ds.$$

Define $\quad Q_\lambda(t) = \sup \{G_\lambda(s) : t - 1 \leq s < \infty\}.$

Then $Q_\lambda(t) \searrow 0$ as $t \to \infty$ and

$$V(t,x(t)) \leq a(|x_o|) - c(\lambda)(t-t_o) + M b^{-1} [\psi(a(r))] Q_\lambda(t_o)(t - t_o + 1),$$

hence

$$V(t,x(t)) \leq [M b^{-1} [\psi(a(r))] Q_\lambda(t_o) - c(\lambda)] (t - t_o) + M b^{-1} [\psi(a(r))] Q_\lambda(t_o) + a(|x_o|) . \tag{4.3.5}$$

Let $0 < \varepsilon < r$. Choose $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$ so that $2a(\delta) < b(\varepsilon)$.

Then choose $T_1(\varepsilon) \geq \tau_\delta + 1$ so that

$$2M b^{-1} [\psi(a(r))] Q_\delta(T_1) < \min [c(\delta), b(\varepsilon)] . \tag{4.3.6}$$

Let $|x_o| < \delta$ and $t_o \geq T_1$. Then we claim

$$|x(t,x_o)| < \varepsilon \text{ on } [t_o,\infty). \tag{4.3.7}$$

Suppose not. Let T_3 be the first point such that $|x(T_3)| = \varepsilon$ and let $T_2 < T_3$ be the last point such that $|x(T_2)| = \delta$. Then $\delta \leq |x(t)| \leq \varepsilon < r$ on $[T_2,T_3]$, hence for $\lambda = \delta$, by (4.3.5) and (4.3.6), we have

$$\begin{aligned} b(\varepsilon) \leq V(T_3,x(T_3)) &\leq [M b^{-1} [\psi(a(r))] Q_\delta(T_2) - c(\delta)] [T_3-T_2] \\ &\quad + M b^{-1} [\psi(a(r))] Q_\delta(T_2) + a(|x(T_2)|) \\ &< M b^{-1} [\psi(a(r))] Q_\delta(T_1) + a(\delta) \\ &< \frac{b(\varepsilon)}{2} + \frac{b(\varepsilon)}{2} = b(\varepsilon), \end{aligned}$$

a contradiction, proving (4.3.7). Now from the continuity of solutions with respect to initial values and the uniqueness of solutions, it follows that the trivial solution of (4.2.2) is uniformly stable if $H(t,0) \equiv 0$.

For the rest of the proof fix $r_1 < r$ and choose $\delta_0 = \delta(r_1)$ and $T_0 = T_1(r_1)$. Fix $t_0 \geq T_0$ and $|x_0| < \delta_0$. Then (4.3.7) implies that $|x(t,x_0)| < r_1$ on $[t_0,\infty)$. Let $0 < \eta < r_1$. Choose $\delta = \delta(\eta)$ and $T_1(\eta)$ as before so that (4.3.6) holds. Choose

$$T = [c(\delta)\ T_1(\eta) + 2\ M\ b^{-1}\ [\psi(a(r))]\ Q_\delta(1) + 2a(r_1)]/c(\delta)$$

$$> T_1(\eta)$$

and it is clear that T depends only on η, not on t_0 or x_0.

Now we claim that

$$|x(t_1,x_0)| < \delta \quad \text{for some } t_1 \text{ in } [t_0 + T_1,\ t_0 + T]. \tag{4.3.8}$$

Suppose not. Then $|x(t,x_0)| \geq \delta$ on $[t_0 + T_1,\ t_0 + T]$. Let $y_0 = x(t_0 + T_1,\ x_0)$. Then by (4.3.5), we have

$$\begin{aligned} 0 < b(\delta) &\leq b(|x(t_0 + T,\ y_0)|) \\ &\leq V(t_0 + T,\ x(t_0 + T)) \\ &\leq [M\ b^{-1}\ [\psi(a(r))]\ Q_\delta(t_0+T_1) - c(\delta)]\ (T-T_1) \\ &\qquad + M\ b^{-1}\ [\psi(a(r))]\ Q_\delta(t_0+T_1) + a(|y_0|) \\ &\leq -\frac{1}{2}\ c(\delta)\ (T-T_1) + M\ b^{-1}\ [\psi(a(r))]\ Q_\delta(1) + a(r_1) \\ &= 0 \end{aligned}$$

a contradiction, proving (4.3.8). Thus by (4.3.7), we obtain

$$|x(t,x(t_1,x_0))| < \eta \quad \text{on } [t_1,\infty)$$

because $t_1 \geq t_0 + T_1 \geq T_1$ and $|x(t_1,x_0)| < \delta$.

Hence $|x(t,x_0)| < \eta$ for $t \geq t_0 + T$.

Since η is arbitrary $|x(t,x_0)| \to 0$ as $t \to \infty$. Further, since T depends only on η and δ depends only on ε, from the continuity of solutions with respect to initial values and the uniqueness of solutions, it follows that the trivial solution of (4.2.2) is uniformly asymptotically stable if $H(t,0) \equiv 0$ and the proof is complete.

Theorem 4.3.2.

Let $F \in \vec{C}_0(x)$ and $F(t,0) = 0$ for all $t \geq 0$.

Assume that

(i) the trivial solution of (4.2.3) is exponentially asymptotically stable,

(ii) $H(t,0) = 0$ for all $t \geq 0$ and for $x \in E$

$$|H(t,x_t(\cdot))| \leq g(t)\,|x(t)|.$$

If $\int_0^\infty g(s)\,ds < \infty$, then the trivial solution of (4.2.2) is exponentially asymptotically stable.

Proof.

Since the trivial solution of (4.2.3) is exponentially asymptotically stable, there exists a Liapunov function V for (4.2.3) on $J \times S_\rho$ satisfying

$$|x| \leq V(t,x) \leq k|x| , \tag{4.3.9}$$

$$V'_{(4.2.3)}(t,x) \leq - \alpha V(t,x), \tag{4.3.10}$$

and

$$|\nabla V(t,x)| \leq M, \tag{4.3.11}$$

where α, k , M are positive constants.

Let $x(t,x_o)$ be a solution of (4.2.2). As in theorem 4.3.1 we have

$$V'_{(4.2.2)}(t,x(t)) \leq V'_{(4.2.3)}(t,x(t)) + \nabla V(t,x(t)) \, . \, H(t,x_t(\cdot)).$$

Now choose $\psi(s) = q^2 s$ where $q > 1$. Using (4.3.10), (4.3.11) and assumption (ii) for $x \in E$, we obtain

$$V'(t,x(t)) \leq - \alpha V(t,x(t)) + M\, q^2 \, g(t) \, V(t,x(t)).$$

Integrating from t_o to t, we have

$$V(t,x(t)) \leq V(t_o,x_o) \exp \left[-\alpha(t-t_o) + Mq^2 \int_{t_o}^{t} g(s) \, ds\right] .$$

Thus the assumptions of the theorem yield the desired result.

Application 4.3.1.

Consider the integro-differential equation (4.2.1).

Assume that

$$|K(t,s,x(s))| \leq L(t,s) \, |x(s)|$$

for $t_o \leq s \leq t < \infty$, $x \in S_\rho$, where $L(t,s)$ is continuous for $t_o \leq s \leq t < \infty$. Choose $\psi(s) = q^2 s$, $q > 1$ and $g(t) = \int_{t_o}^{t} L(t,s) \, ds$.

Then for $x \in E$, the conclusion of theorem 4.3.2 remains valid.

Let $V(t,x)$ be a real valued function continuous in (t,x) for $t \geq t_0$ and $x \in S_\rho$.

We define

$$D_- V(t,x(t)) = \liminf_{h \to 0^-} \frac{1}{h} [V(t+h, x(t)+h\ [F(t,x(t))+H(t,x_t(\cdot))]) - V(t,x(t))]$$

and let $E_A = \{x : J \to R^n : x$ is continuous and

$$V(s,x(s))\ A(s) \leq V(t,x(t))\ A(t),\ t_0 \leq s \leq t\}.$$

Theorem 4.3.3.

Assume that there exist functions $V(t,x)$ and $A(t)$ satisfying the following properties :

(i) $A(t) > 0$ is continuous for $t \in J$ and $A(t) \to \infty$ as $t \to \infty$,

(ii) $V(t,x)$ is defined and continuous on $J \times S_\rho$ with values in $[0, \infty)$, and $V(t,0) = 0$. $V(t,x)$ is positive definite and decrescent and locally Lipschitzian in x,

(iii) $A(t)\ D_-\ V(t,x(t)) + V(t,x(t))\ D_-\ A(t) \leq 0$ for $t \geq t_0$ and $x \in E_A$.

Then, the trivial solution of (4.2.2) is uniformly asymptotically stable.

The proof of the theorem is similar to that of theorem 3.2 in [24] and hence omitted.

Application 4.3.2.

Consider a special case of (4.2.1)

$$x'(t) = A\ x(t) + \int_{t_0}^{t} K(t,s)\ x(s)\ ds \tag{4.3.12}$$

where A is an $n \times n$ real constant matrix, the eigen values of which have negative real parts, $K(t,s)$ is a continuous $n \times n$ matrix for $t_0 \leq s \leq t < \infty$ and $x \in R^n$.

It is easy to see that there exists a positive definite, symmetric real matrix B such that $BA + A^T B = -I$, where I is the identity matrix, A^T the transpose of A. Let λ and Λ be the least and greatest eigen values of B respectively. Then we have $\lambda > 0$ and

$$\lambda^2(x,x) \leq (x,Bx) \leq \Lambda^2(x,x)$$

for all $x \in R^n$.

Take $L(t,x) = A(t)\, V(t,x) = e^{\alpha t}\,(x,Bx)$, $\alpha > 0$.

Using the relation $A^T B + BA = -I$ and (4.3.12), we obtain

$$L'(t,x(t)) = -\,e^{\alpha t}\,|x(t)|^2 + 2\,e^{\alpha t}\left(\int_{t_0}^{t} K(t,s)x(s)ds, Bx(t)\right) + \alpha\,e^{\alpha t}\,(x(t), Bx(t)). \qquad (4.3.13)$$

The subset E_A is given by

$$E_A = \{x : \lambda^2\, x^2(s)\, e^{\alpha s} \leq \Lambda^2\, x^2(t)\, e^{\alpha t},\ t \geq s \geq t_0\}.$$

Then for $x \in E_A$ we have $|x(s)| \leq \frac{\Lambda}{\lambda}\, e^{\frac{\alpha}{2}(t-s)}\, |x(t)|$.

Therefore

$$\left(\int_{t_0}^{t} K(t,s)x(s)ds,\ Bx(t)\right) \leq \frac{\Lambda}{\lambda}\,|B|\,|x(t)|^2 \int_{t_0}^{t} e^{\frac{\alpha}{2}(t-s)}\,|K(t,s)|\,ds.$$

Hence from (4.3.13) we have

$$L'(t,x(t)) \leq e^{\alpha t}\,|x(t)|^2\left[-1+2\,|B|\,\frac{\Lambda}{\lambda}\int_{t_0}^{t} e^{\frac{\alpha}{2}(t-s)}\,|K(t,s)|\,ds + \alpha\Lambda^2\right].$$

Thus the application of theorem 4.3.3 yields the desired result provided

$$\int_{t_0}^{t} e^{\frac{\alpha}{2}(t-s)} |K(t,s)| \, ds \leq \frac{(1-\alpha\Lambda^2)}{2\,|B|\,\Lambda} \tag{4.3.14}$$

holds. Since α is arbitrary, letting $\alpha \to 0$, the condition (4.3.14) reduces to

$$\int_{t_0}^{t} |K(t,s)| \, ds \leq \frac{\lambda}{2\,|B|\,\Lambda} \tag{4.3.15}$$

which is a sufficient condition for uniform stability of the trivial solution of (4.3.12).

Remark 4.3.1.

Condition (4.3.15) and for each $T > 0$

$$\lim_{t \to \infty} \int_{t_0}^{T} |K(t,s)| \, ds = 0 \tag{4.3.16}$$

have been extensively used by many authors (cf. [36]) for asymptotic stability of the trivial solution of nonlinear integral equations.

Example 4.3.1.

Choose $K(t,s) = e^{-\frac{\alpha}{2}(t+s)}$, $\alpha > 0$

for $0 \leq s \leq t$. We see that the conditions (4.3.14), (4.3.15) and (4.3.16) are satisfied.

Example 4.3.2.

Consider the scalar integro-differential equation

$$x'(t) = -\frac{1}{3}x(t) + \int_{0}^{t} e^{-3(t-s)} x(s) \, ds \tag{4.3.17}$$

for $0 \leq s \leq t < \infty$. Then obviously $B = 3/2$ and $\lambda = \Lambda = 3/2$. Choose $L(t,x) = e^{\alpha t} V(t,x) = e^{4t} (x,Bx)$. It is clear that the condition (4.3.14) is not satisfied and hence the trivial solution of (4.3.17) is not uniformly asymptotically stable. On the other hand the condition (4.3.15) is satisfied and thus it is uniformly stable. Indeed, our conclusion is supported by the fact that the solutions of (4.3.17) are given by $x(t) = \frac{x(0)}{10} [e^{-10t/3} + 9]$.

Example 4.3.3. (cf. [16]).

Consider the scalar differential equation

$$x'(t) = -2x(t) + x(0). \tag{4.3.18}$$

Equation (4.3.18) may be treated as a perturbation to the equation

$$x'(t) = - 2x(t).$$

Choose $\quad V(x(t)) = \frac{x^2(t)}{2}$ and $\psi(s) = 2s$.

Then $\quad V'(x(t)) \leq -4V(x(t)) + 2\sqrt{2}\; V(x(t))$.

Here $g(t) = 2\sqrt{2}$. Obviously $\int_t^{t+1} g(s)\, ds \not\to 0$ as $t \to \infty$. Thus all the conditions of the theorem 4.3.1 are not satisfied. Furthermore, it is clear that the trivial solution of (4.3.18) is not uniformly asymptotically stable.

4.4 Integro-differential Equations Containing Measures

Consider the following integro-differential equation

$$Dx = F(t,x) + G(t,x)\, Du + \int_0^t K(t,s,x(s))\, ds \tag{4.4.1}$$

where F, $G : J \times R^n \to R^n$, $K : J \times J \times R^n \to R^n$ and $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J. v_u denotes the total variation function of u.

Assuming all smooth conditions which ensure the existence of a unique solution $x(t) = x(t,t_0,x_0)$ of (4.4.1), we state a theorem that gives the eventual uniform asymptotic stability property of the zero solution of (4.4.1).

Theorem 4.4.1.

Let $F \in \bar{C}_0(x)$ and $F(t,0) = 0$ for all $t \geq 0$.

Assume that

(i) the trivial solution of (4.2.3) is eventually uniformly asymptotically stable.

(ii) $G(t,x)$ and $K(t,s,x)$ satisfy the following hypothesis : there exists $r > 0$ such that for every λ, $0 < \lambda < r$, there exist $\tau_\lambda \geq 0$, a measurable function $g_\lambda(t)$ defined on $[\tau_\lambda,\infty)$ and an integrable function $h_\lambda(t,s)$ on $\tau_\lambda \leq s \leq t < \infty$ such that

$$|G(t,x)| \leq g_\lambda(t)$$

and $|K(t,s,x)| \leq h_\lambda(t,s)$ for all $\lambda \leq |x| < r$ and $t \geq \tau_\lambda$,

$$\text{where } G_\lambda(t) = \int_t^{t+1} g_\lambda(s)\, dv_u(s) \to 0 \text{ as } t \to \infty \tag{4.4.2}$$

$$\text{and } H_\lambda(t) = \int_t^{t+1} \left(\int_{\tau_\lambda}^{s} h_\lambda(s,\tau)\, d\tau\right) ds \to 0 \text{ as } t \to \infty \tag{4.4.3}$$

(iii) the discontinuous points $t_0 < t_1 < t_2 < t_3 < \dots < t_k < \dots$ of u tend to ∞ as k tends to ∞.

Then there exist $T_0 \geq 0$ and $\delta_0 > 0$ such that if $t_0 \geq T_0$, $|x_0| < \delta_0$ then the solution $x(t) = x(t,t_0,x_0)$ of (4.4.1) satisfies $|x(t)| \to 0$ as $t \to \infty$. In particular if $G(t,0) = 0$ and $K(t,s,0) = 0$ for all $t_0 \leq s \leq t < \infty$, then the trivial solution of (4.4.1) is eventually uniformly asymptotically stable.

The proof of this theorem runs exactly on the same lines of that of theorem 3.5.1. Indeed it is a combination of the arguments given in the proofs of theorem 3.5.1 and corollary 5.1 of [46].

It is clear that the equation (4.4.1) is treated as a perturbed system of (4.2.3) involving two perturbations, one is impulsive in character and the other is continuous in nature. Notice that the equation (4.4.1) with $K \equiv 0$ is studied in section 3.5 and sufficient conditions are obtained for eventual uniform asymptotic stability of the trivial solution of (4.4.1) (theorem 3.5.1), while in section 4.3 the equation (4.4.1) with $G \equiv 0$ is discussed by using Liapunov-Razumikhin conditions (theorem 4.3.1). Observe that the conditions (4.4.2) and (4.4.3) are respectively the condition (H_2) of theorem 3.5.1 and the assumption (iii) of corollary 5.1 in [46]. Authors of [46] have proved the uniform asymptotic stability of the trivial solution of (4.4.1) with $G \equiv 0$, assuming the uniform asymptotic stability of (4.2.3). It is surprising to note that the condition (4.4.3) on K together with the condition (4.4.2) gives only eventual uniform asymptotic stability of the trivial solution of (4.4.1), while the trivial solution of (4.2.3) is

uniformly asymptotically stable. Although the perturbation $\int_0^t K(t,s,x(s))\,ds$ is continuous, it is not improving the stability behavior of (4.4.1). This naturally poses the following question "under what conditions on G and K, uniform asymptotic stability of the trivial solution of (4.2.3) implies the uniform asymptotic stability of the trivial solution of (4.4.1)". The corresponding conditions on G and K would be interesting and important as they indicate the effect of more than one perturbation acting on the system directly or indirectly.

CHAPTER - 5

NONLINEAR MIXED INTEGRAL EQUATIONS INVOLVING LEBESGUE-STIELTJES INTEGRALS

5.1 Introduction

Bihari [5] , Miller, Nohel and Wong [37] , Nashed and Wong [39] among others have studied integral equations of the mixed type such as Volterra-Fredholm, Volterra-Hammerstein and Volterra-Urysohn, using fixed point theorems and many interesting results have been accumulated. Integral equations of mixed type arise naturally in a number of problems in ordinary differential equations, in particular, certain class of singular perturbation problems, boundary value problems on an infinite half line and also in perturbed operator equations. In many problems of physics and engineering (optimal control theory in particular) one can not expect the perturbations to be well behaved and therefore important to consider the cases when the perturbations are impulsive (cf. [13] , [42] , [44] , [45] , [54]). Such systems would be described by differential equations containing measures which are equivalent to integral equations of mixed type involving Lebesgue-Stieltjes integrals.

This chapter is mainly concerned with the questions of existence, uniqueness and stability properties of solutions of integral equations of Volterra-Fredholm type involving Lebesgue-Stieltjes integrals, namely

$$x(t) = f(t) + \int_0^t a(t,s)g(s,x(s))du(s) + \int_0^\infty b(t,s)h(s,x(s))dw(s) \quad (5.1.1)$$

where x, f, g, h are vectors with n-components, a and b are $n \times n$ matrices, u and w are right continuous functions of bounded variation

on every compact subinterval of $J = [0,\infty)$. The results of this chapter improve and include some of the results of the above mentioned authors and also the study of the corresponding results for differential systems with impulsive perturbations, Volterra integral equations with discontinuous perturbations and boundary value problems on an infinite half line.

5.2 Existence, Uniqueness and Stability

We require the following conditions for our subsequent discussion :

$(\bar{H}_1)$ u and w are right continuous real valued functions defined on J and are functions of bounded variation on every compact subinterval of J.

$(\bar{H}_2)$ $a(t,s)$ is integrable with respect to u, for a fixed t,

$$\Phi : 0 = t_0 < t_1 < \dots < t_n = t \quad \text{and}$$

$$\sup_{t \geq 0} \sup_{\Phi} \left\{ \sum_{i=1}^{n} \left[\int_{t_0}^{t_{i-1}} |a(t_i,s) - a(t_{i-1},s)| \, dU(s) + \int_{t_{i-1}}^{t_i} |a(t_i,s)| \, dU(s) \right] \right\} \leq A_0 < \infty$$

where A_0 is a positive constant and $U(s) = \overset{s}{\underset{0}{V}} u(s)$.

$(\bar{H}_3)$ $b(t,s)$ is integrable with respect to w and

$$\sup_{t \geq 0} \sup_{\Phi} \left\{ \sum_{i=1}^{n} \left[\int_{0}^{\infty} |b(t_i,s) - b(t_{i-1},s)| \, dW(s) \right] \right\} \leq B_0 < \infty$$

where $W(s) = \overset{s}{\underset{0}{V}} w(s)$ and B_0 is a positive constant.

Theorem 5.2.1. Let $(\bar{H}_1)$, $(\bar{H}_2)$ and $(\bar{H}_3)$ be satisfied. Further assume that g and h satisfy the following hypotheses :

(i) $g(t,x)$ is integrable (in the Lebesgue-Stieltjes sense) with respect to u and $g(t,0) \equiv 0$.

(ii) For each $\gamma > 0$, there exists a $\delta > 0$ such that

$$|g(t,x) - g(t,y)| \leq \gamma|x - y| \text{ for all } |x|, |y| \leq \delta$$

and uniformly in t.

(iii) $h(t,x)$ is integrable (in the Lebesgue-Stieltjes sense) with respect to w and $h(t,0) \equiv 0$.

(iv) For each $\xi > 0$, there exists a $\eta > 0$ such that

$$|h(t,x) - h(t,y)| \leq \xi |x - y| \text{ for all } |x|,|y| \leq \eta$$

and uniformly in t.

(v) $a(t,s)\, g(t,x)$ and $b(t,s)\, h(t,x)$ are integrable with respect to u and w respectively.

Then there exists a number $\varepsilon_0 > 0$ such that to any $\varepsilon_1 \in (0,\varepsilon_0]$ there corresponds a $\delta_1 > 0$ such that for $||f|| \leq \delta_1$, there exists a unique solution $x(t)$ of (5.1.1) on $0 \leq t < \infty$ satisfying $||x|| \leq \varepsilon_1$.

Proof.

Fix $\xi > 0$ such that $\xi B_0 < 1$. By the assumptions (iii) and (iv) of the theorem we have a $\eta > 0$ such that $|h(t,x)| \leq \xi|x|$ uniformly in t whenever $|x| \leq \eta$. Let $\gamma = \dfrac{1 - \xi B_0}{2A_0}$ and choose a $\delta > 0$ such that $|g(t,x_1) - g(t,x_2)| \leq \gamma|x_1 - x_2|$, uniformly in $t \geq 0$ whenever $|x_1|,|x_2| \leq \delta$. Let $\varepsilon_0 = \min(\eta,\delta)$.

For any $\varepsilon_1 > 0$, $0 < \varepsilon_1 \leq \varepsilon_0$, define the set

$$S(\varepsilon_1) = \{x \in BV(J) : ||x|| \leq \varepsilon_1\}.$$

For $x \in S(\varepsilon_1)$ define an operator T on $S(\varepsilon_1)$ as follows :

$$(Tx)(t) = f(t) + \int_0^t a(t,s)g(s,x(s))\, du(s) + \int_0^\infty b(t,s)h(s,x(s))\, dw(s).$$

For $x \in S(\varepsilon_1)$ and using the assumptions (i) - (iv) of the theorem, we obtain

$$||Tx|| = ||f(t) + \int_0^t a(t,s)g(s,x(s))du(s) + \int_0^\infty b(t,s)h(s,x(s))\, dw(s)||$$

$$\leq ||f|| + ||\int_0^t a(t,s)g(s,x(s))\, du(s)|| + ||\int_0^\infty b(t,s)\, h(s,x(s))dw(s)||$$

$$\leq ||f|| + \sup_{t \geq 0} \sup_{\Phi} \{ \sum_{i=1}^n [\int_{t_0}^{t_{i-1}} |a(t_i,s) - a(t_{i-1},s)| \gamma |x(s)|\, dU(s)$$

$$+ \int_{t_{i-1}}^{t_i} |a(t_i,s)|\, \gamma |x(s)|\, dU(s)] \}$$

$$+ \sup_{t \geq 0} \sup_{\Phi} \{ \sum_{i=1}^n [\int_0^\infty |b(t_i,s)-b(t_{i-1},s)|\, \xi |x(s)|\, dW(s)] \}.$$

Therefore by $(\bar{H}_2)$ and $(\bar{H}_3)$ we obtain,

$||Tx|| \leq \delta_1 + \gamma A_0\, \varepsilon_1 + \xi B_0\, \varepsilon_1 < \varepsilon_1$ provided $\delta_1 < (1 - \xi B_0)(\varepsilon_{1/2})$.

This shows that $TS(\varepsilon_1) \subset S(\varepsilon_1)$.

Let $x,y \in S(\varepsilon_1)$

$$||Tx - Ty|| = ||\int_0^t a(t,s)\, [g(s,x(s)) - g(s,y(s))]\, du(s)$$

$$+ \int_0^\infty b(t,s)\, [h(s,x(s)) - h(s,y(s))]\, dw(s)||.$$

By using $(\bar{H}_2)$, $(\bar{H}_3)$ and the assumptions (ii) and (iv) of the theorem, we have

$$||Tx - Ty|| \leq (\gamma A_o + \xi B_o)\, ||x - y|| = \frac{1 + \xi B_o}{2}\, ||x - y||.$$

Since $0 < \xi B_o < 1$, T is indeed a contraction on $S(\varepsilon_1)$. Hence the application of Banach fixed point theorem yields the desired result.

Corollary 5.2.1.

Under the assumptions of the theorem 5.2.1 if in addition $f(t) \to 0$ as $t \to \infty$ and for each $T_1 > 0$

$$(\bar{H}_4) \qquad \lim_{t\to\infty} \int_0^{T_1} |a(t,s)|\, dU(s) = 0$$

and

$$(\bar{H}_5) \qquad \lim_{t\to\infty} \int_0^{T_1} |b(t,s)|\, dW(s) = 0$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof.

Define the set $S_o(\varepsilon_1) = \{x \in S(\varepsilon_1) : \lim_{t\to\infty} x(t) = 0\}$.

Clearly $S_o(\varepsilon_1)$ is a closed subspace of $S(\varepsilon_1)$ under the norm $||\cdot||$. We need only to prove, that $(Tx)(t) \to 0$ as $t \to \infty$ for $x \in S_o(\varepsilon_1)$ since the rest of the proof is similar to that of theorem 5.2.1. Let $\eta > 0$ be given. By hypotheses, choose $T_1 > 0$ such that $|x(t)| \leq \eta/5$, $|f(t)| \leq \eta/5$ for all $t \geq T_1$. By $(\bar{H}_4)$ and $(\bar{H}_5)$ we can find a $T_2 \geq T_1$ so large that $\int_0^{T_1} |a(t,s)|\, dU(s) \leq \frac{\eta A_o}{5\varepsilon_1}$ and $\int_0^{T_1} |b(t,s)|\, dW(s) \leq \frac{\eta B_o}{5\varepsilon_1}$ for

$t \geq T_2$. Then for $t \geq T_2$, we observe that

$$|(Tx)(t)| \leq |f(t)| + \int_0^t |a(t,s)|\ |g(s,x(s))|\ dU(s)$$

$$+ \int_0^\infty |b(t,s)|\ |h(s,x(s))|\ dW(s)$$

$$\leq |f(t)| + \int_0^t |a(t,s)|\ \gamma|x(s)|\ dU(s) + \int_0^\infty |b(t,s)|\ \xi|x(s)|\ dW(s)$$

$$\leq \frac{\eta}{5} + \int_0^{T_1} |a(t,s)|\ \gamma|x(s)|\ dU(s) + \int_{T_1}^t |a(t,s)|\ \gamma|x(s)|\ dU(s)$$

$$+ \int_0^{T_1} |b(t,s)|\xi|x(s)|\ dW(s) + \int_{T_1}^\infty |b(t,s)|\xi\ |x(s)|\ dW(s).$$

$$\leq \frac{\eta}{5} + \frac{\eta A_0}{5\epsilon_1}\gamma\epsilon_1 + \frac{\eta}{5}\ \gamma A_0 + \frac{\eta B_0}{5\epsilon_1}\xi\ \epsilon_1 + \frac{\eta}{5}\xi B_0$$

$< \eta$, since $\xi B_0 < 1$ by the choice of ξ in theorem 5.2.1.

Since $\eta > 0$ is arbitrary, $(Tx)(t) \to 0$ as $t \to \infty$. Thus $T\ S_0(\varepsilon_1) \subset S_0(\varepsilon_1)$. This completes the proof.

Remark 5.2.1.

Theorem 5.2.1 may be regarded as a stability result for the system (5.1.1) in the following sense. Given any $\varepsilon_1 > 0$ sufficiently small, there exists a $\delta_1 > 0$ such that for every $f \in BV(J)$, $||f|| \leq \delta_1$ the solution $x(t)$ of (5.1.1) is in $BV(J)$ and $||x|| \leq \varepsilon_1$. Furthermore in corollary 5.2.1, if $f \in BV(J)$ and $f(t) \to 0$ as $t \to \infty$, then $x(t) \to 0$ as $t \to \infty$. Thus corollary 5.2.1 is a type of asymptotic stability theorem for the system (5.1.1).

5.3 General Discussion

The system (5.1.1) can also be considered as a generalization of the system of differential equations

$$\frac{dy}{dt} = A(t)\, y \tag{5.3.1}$$

and

$$Dx = A(t)\, x + G(t,x)\, Du \tag{5.3.2}$$

where $x,\ y \in R^n$, $A(t)$ is an $n \times n$ matrix, Du denotes the distributional derivative of the function u. $G : J \times R^n \to R^n$, and $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J. Here Du can be identified with Stieltjes measure and has the effect of suddenly changing the state of the system at the points of discontinuity of u. Indeed, if we denote by $\Phi(t)$ the matrix function satisfying the matrix differential equation $\dot{\Phi}(t) = A(t)\, \Phi(t)$ with $\Phi(0) = I$, $\Phi \in C^\infty(J)$, then the solution $x(t)$ of of (5.3.2) (cf. [44]) can be written as

$$x(t) = \Phi(t)\, x_0 + \int_0^t \Phi(t)\, \Phi^{-1}(s)\, G(s,x(s))\, du(s). \tag{5.3.3}$$

If we are interested in the existence of bounded solutions of (5.3.2), then we must consider the equation

$$x(t) = \Phi(t)\, x_0 + \int_0^t \Phi(t)\, P_0\, \Phi^{-1}(s)\, G(s,x(s))\, du(s)$$
$$- \int_t^\infty \Phi(t)\, P_1\, \Phi^{-1}(s)\, G(s,x(s))\, du(s) \tag{5.3.4}$$

where P_0 is the projector of R^n on the subspace X_0 of the values at $t = 0$ of the bounded solutions of (5.3.1), and P_1 is the projector

from R^n on the subspace X_1, such that $R^n = X_o \oplus X_1$. We shall choose always $x_o \in X_o$.

The system (5.3.4) has recently been studied by Bihari [5] by similar techniques but with bounded kernels, where u is an absolutely continuous function on J.

Example 5.3.1 (when $n = 1$).

Choose $a(t,s) = \dfrac{1}{(1+s)^2}$ for $0 \leq s \leq t < \infty$

and $b(t,s) = e^{-(t+s)}$ for $0 \leq s,\ t < \infty$.

Select

$$u(s) = \begin{cases} 0 & \text{for } 0 \leq s < \alpha \\ s & \text{for } \alpha \leq s \leq t \end{cases}$$

$$\text{and}\quad w(s) = \begin{cases} \beta e^{-s} & \text{for } 0 \leq s < \alpha \\ \beta e^{-2s} & \text{for } \alpha \leq s \leq t, \end{cases}$$

where α, β are fixed positive numbers. Then we see that the hypotheses $(\bar{\bar{H}}_2)$ and $(\bar{\bar{H}}_3)$ are satisfied.

Example 5.3.2 (when $n = 1$).

Choose $a(t,s) \equiv 1$ for $0 \leq s \leq t < \infty$ and

select

$$u(s) = \begin{cases} e^{-s} & \text{for } 0 \leq s < \alpha \\ e^{-2s} & \text{for } \alpha \leq s \leq t, \end{cases}$$

where α is a fixed positive number. The hypothesis $(\bar{\bar{H}}_2)$ can be easily verified. Moreover the

hypothesis $(\bar{\bar{H}}_3)$ holds if $b(t,s) = e^{(\bar{e}^t + \bar{e}^s - s)}$ for $0 \leq s,\ t < \infty$ and $w(s)$ as in example 5.3.1.

It is apt to remark here that the hypothesis $(\bar{\bar{H}}_2)$ implies that

$$\sup_{t \geq 0} V\left(\int_0^t |a(t,s)|\, dU(s),\ [0,t]\right) \leq A_0 \qquad (H_2^*)$$

and the hypothesis $(\bar{\bar{H}}_3)$ implies that

$$\sup_{t \geq 0} V\left(\int_0^\infty |b(t,s)|\, dW(s),\ [0,t]\right) \leq B_0 \qquad (H_3^*)$$

but however the converses are not true as the following example shows.

Example 5.3.3 (when $n = 1$).

Select $a(t,s) = e^{-(t-s)}$, $0 \leq s \leq t < \infty$ and $u(s) = s$. Clearly, the condition (H_2^*) is satisfied but not condition $(\bar{\bar{H}}_2)$. However, the condition $(\bar{\bar{H}}_2)$ implies (H_2^*).

Note that the hypotheses $(\bar{\bar{H}}_2)$ and $(\bar{\bar{H}}_3)$ are some what restrictive. For example, $(\bar{\bar{H}}_2)$ does not include the case where $a(t,s) = a(t-s)$ with $a \in L^1(0,\infty)$ and $u(s) = s$ (see example 5.3.3). Even the choice of

$$a(\tau) = \begin{cases} 1 & \text{on } 0 \leq \tau \leq 1 \\ 0 & \text{on } \tau > 1,\ u(s) = s \end{cases}$$

fails to satisfy $(\bar{\bar{H}}_2)$. However the condition (H_2^*) holds.

Similarly $(\bar{\bar{H}}_3)$ does not include the case of a kernel of convolution type which is $L^1(-\infty,\infty)$. Thus the results of this chapter pose the

following problem. "Is the theorem 5.2.1 true if we replace $(\bar{\bar{H}}_2)$ and $(\bar{\bar{H}}_3)$ by (H_2^*) and (H_3^*) respectively?" If the answer is affirmative, then the corresponding results improve and include some of the results of [35] , [37] and [40] .

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